Journal
of
The Franklin Institute
Devoted to Science and the Mechanic Arts

Vol. 271 FEBRUARY, 1961 No. 2

ON THE SUMMATION OF $\sum_{n=0}^{\infty} \frac{f_n}{n^k}$ AND ITS ASSOCIATED INTEGRALS *

BY

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SUMMARY

A geometrical interpretation of the convolution z transform is given and this is applied to develop z transforms of functions of the type $f(t)/t^k$, $k > 0$. After establishing the equivalence between certain forms of integrals, the authors have used the integrals to sum certain forms of infinite series. Possible extensions of this technique to other forms of series and to sum finite series are also indicated.

INTRODUCTION

z transformation technique has been successfully applied to solve linear difference equations. Convolution of z transform and modified z transform developed by Cypkin (1)[2] and one of the authors (2), respectively, will be used in this paper to derive the z transform of certain discrete time functions of the type $(f_n)/(n^k)$. These z transforms will then be utilized to sum infinite series, which are well known in mathematical literature. During the discussion equivalence between certain types of integrals and their use will be established. First the complex convolution of two discrete functions will be explained constituting a brief review of (2), and the paths of integration will be defined. Their importance will be evident later.

1. COMPLEX CONVOLUTION (1, 2)

Suppose we have two continuous time functions $f(t)$ and $h(t)$ having $F(s)$ and $H(s)$ as their Laplace transforms. Let $f^*(t)$ and $h^*(t)$ be the

* This research was supported by the U. S. Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract No. AF 18(600)-1521. Reproduction in whole or in part is permitted for any purpose of the United States Government.

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

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sampled functions, their Laplace transforms being $F^*(s)$ and $H^*(s)$. It is required to find the Laplace transform of $[f^*(t)h^*(t)]$.

Using complex convolution in the s plane (2, 3) we have

$$L[f^*(t)h^*(t)] = \frac{T}{2\pi j}\int_{c-j(\pi/T)}^{c+j(\pi/T)} F^*(q)H^*(s-q)dq. \tag{1}$$

Both $F^*(s)$ and $H^*(s)$ have an infinite number of poles in the s plane because of the sampling action. Assuming as an example that $F(s)$ has a pole in the left hand plane at $s = s_\alpha$, and $H(s)$ has a pole also in the left hand plane at $s = s_\beta$ (both s_α and s_β real), then the pole-zero pattern in both the s and q planes of $F^*(s)$ and $H^*(s)$ is as shown in Figs. 1, 2.[3]

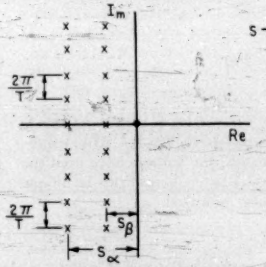

FIG. 1. Pole-zero pattern of $F^*(s)$ and $H^*(s)$ in s plane.

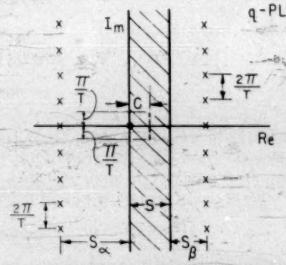

FIG. 2. Region of analyticity in the q plane and path of integration.

The strip of analyticity in the q plane is shown cross-hatched and the path of integration is shown dotted. Since both $F^*(s)$ and $H^*(s)$ are functions of e^{Ts} only, we can replace e^{Ts} by z and e^{Tq} by p so that $F^*(q)$ becomes a function of p only and $H^*(s-q)$ becomes a function of z/p only. As required by the mapping $dq = (1/(Te^{Tq}))dp = ((p^{-1})/T)dp$. The mapping of the s plane into the z plane and q plane into the p plane should be obvious by referring to Figs. 3 and 4. Note that the strip of analyticity in the q plane maps into an annulus of convergence into the p plane. With these substitutions

$$L[f^*(t)h^*(t)] \to \mathfrak{z}[f(t)h(t)] = \frac{1}{2\pi j}\int_\Gamma p^{-1}F^*(p)H^*\left(\frac{z}{p}\right)dp. \tag{2}$$

where Γ is the dotted contour in the p plane.

By Cauchy's theorem on residues we can evaluate residues at the singularities of $p^{-1}F^*(p)$. If the value of the integral along the infinite circle tends to 0 then we can also evaluate residues at the singularities

[3] Equation 1 is applicable even if $F(s)$ and $H(s)$ have poles in the right-hand plane.

(2) of $H^*\left(\frac{z}{p}\right)$. Therefore

$$\begin{aligned}\mathfrak{z}[f(t)h(t)] &= \frac{1}{2\pi j}\int_\Gamma p^{-1}F^*(p)H^*\left(\frac{z}{p}\right)dp \\ &= \sum \text{Residues at singularities of } p^{-1}F^*(p) \qquad (3) \\ &= -\sum \text{Residues at singularities of } H^*\left(\frac{z}{p}\right).\end{aligned}$$

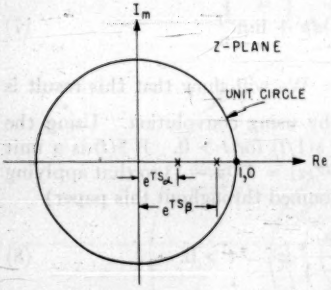

FIG. 3. Poles of $F^*(z)$ and $H^*(z)$ in z plane.

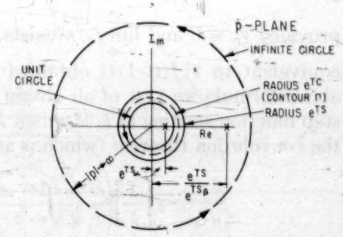

FIG. 4. Path of integration in p plane.

This can be illustrated by a simple example.

Consider:

$$F^*(z) = \frac{z}{z-a} \qquad H^*(z) = \frac{z}{z-b}.$$

Now

$$\begin{aligned}\mathfrak{z}[f(t)h(t)] &= \frac{1}{2\pi j}\int_\Gamma \frac{p^{-1}p}{p-a}\frac{z}{z-pb}\,dp \\ &= \frac{1}{2\pi j}\int_\Gamma -\frac{z}{b(p-a)\left(p-\frac{z}{b}\right)}\,dp. \qquad (4)\end{aligned}$$

Denominator of the integrand is at least two degrees higher than the numerator in p. Hence the value of the integral along infinite circle is zero.

Now evaluating residues inside the contour Γ, we have

$$\mathfrak{z}[(f(t)h(t))] = \frac{z}{z-ab}. \qquad (5)$$

Evaluating residues with respect to singularities of $H^*\left(\frac{z}{p}\right)$, we have

$$\mathfrak{z}[f(t)h(t)] = -\left(-\frac{z}{b\left(\frac{z}{b}-a\right)}\right) = \frac{z}{z-ab}. \tag{6}$$

2. z TRANSFORM OF $f(t)/t^k$ (k IS AN INTEGER > 0)

The z transform of $f(t)/t$ has been derived in a previous paper (**2**) as

$$\frac{1}{T}\int_z^\infty p^{-1}F^*(p)dp + \lim_{n\to 0}\frac{f_n}{n} \tag{7}$$

provided $f_n = 0$ and $\lim_{n\to 0} f_n/n$ exists. We will show that this result is equivalent to $\mathfrak{z}\{f(t)\cdot 1/t\}$ obtained by using convolution. Using the above formula we first of all obtain $\mathfrak{z}(1/t)$ for $t > 0$. If $f(t)$ is a unit step function starting at $t > 0$ then $F^*(z) = 1/(z-1)$ so that applying the convolution theorem (which is assumed throughout this paper)

$$\mathfrak{z}\left[\frac{1}{t}\right] = \frac{1}{T}\int_z^\infty \frac{z^{-1}}{z-1}dz \qquad t > 0. \tag{8}$$

Choosing $T = 1$ we have

$$\mathfrak{z}\left(\frac{1}{t}\right) = \ln\frac{z}{z-1} \qquad t > 0. \tag{9}$$

For any $f(t)$ we have

$$\mathfrak{z}\frac{f(t)}{t} = \frac{1}{2\pi j}\int_\Gamma p^{-1}F^*(p)\ln\frac{z}{z-p}dp + \lim_{n\to 0}\frac{f_n}{n}$$

$$= \int_z^\infty p^{-1}F^*(p)dp + \lim_{n\to 0}\frac{f_n}{n}. \tag{10}$$

We thus have two ways of evaluating the z transform of $f(t)/t$.[4] We can establish the above result quite independently by considering the convolution integral in the p plane. $\ln z/(z-p)$ has a branch cut in the p plane extending from $p = z$ to $p = \infty$. Since we are free to take the branch cut anywhere in the p plane, we will take it along the positive real axis. This is illustrated in Fig. 5. As before, the cross-hatched

[4] This equivalence should be utilized advantageously in evaluating the z transform of $f(t)/t$ and also in summing infinite series.

area shows annulus of convergence and we have

$$\int_{C_1} = -\int_{EA}$$

$$\int_{EA} + \int_{AB} + \int_{BCD} + \int_{DE} = 0$$

$$-\int_{EA} = \int_{AB} + \int_{DE} + \int_{BCD}$$

$$\int_{C_1} = \int_{AB} + \int_{DE} + \int_{BCD}.$$

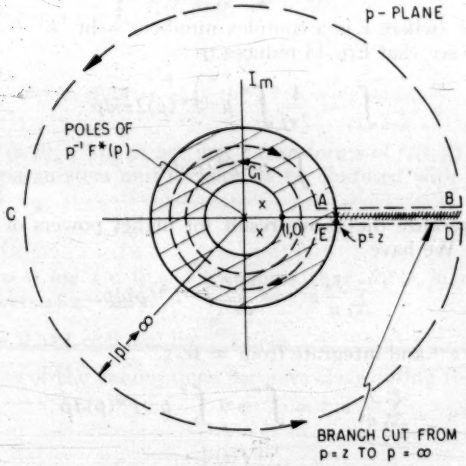

FIG. 5. Branch cut in p plane and path of integration.

Along AB, $p = |p|e^{i\phi}$ $\quad \phi \to 0$
Along DE, $p = |p|e^{i(2\pi-\phi)}$ $\quad \phi \to 0$

$$\ln \frac{z}{z-p} = \ln\left(-\frac{z}{p}\right) + \ln \frac{1}{1-\frac{z}{p}} \tag{11}$$

Expansion for $\ln \dfrac{1}{1-\dfrac{z}{p}}$ is given by (4)

$$\ln \frac{1}{1-\frac{z}{p}} = \frac{z}{p} + \frac{z^2}{2p^2} + \cdots \qquad \left(\left|\frac{z}{p}\right| < 1.\right) \tag{12}$$

Therefore

$$\ln \frac{z}{z-p} = \ln\left(-\frac{z}{p}\right) + \frac{z}{p} + \frac{z^2}{2p^2} + \cdots. \tag{13}$$

Now $\int_{BCD} = 0$. Hence,

$$\int_{C_1} = \frac{1}{2\pi j}\int_z^\infty \left[p^{-1}F^*(p)\left\{\ln\left(-\frac{z}{p}\right) + \frac{z}{p} + \frac{z^2}{2p^2}\cdots\right\}\right]\Big|_{p=|p|e^{j\phi}\to\infty} dp$$

$$-\frac{1}{2\pi j}\int_z^\infty \left[p^{-1}F^*(p)\left\{\ln\left(-\frac{z}{p}\right) + \frac{z}{p} + \frac{z^2}{2p^2} + \cdots\right\}\right]\Big|_{p=|p|e^{j(2\pi-\phi)}\to\infty} dp. \tag{14}$$

Now $\ln k$ (where k is a complex number) $= \ln|k| + \arg k$. It is then easy to see that Eq. 14 reduces to

$$\int_{C_1} = \frac{1}{2\pi j}\int_z^\infty p^{-1}F^*(p)2\pi j dp$$

$$= \int_z^\infty p^{-1}F^*(p)dp. \tag{15}$$

One can generalize the above result for higher powers of t in the denominator. We have

$$\sum_{n=1}^{\infty}\frac{f_n}{n}z^{-n} = \int_z^\infty p^{-1}F^*(p)dp.$$

Multiply by z^{-1} and integrate from ∞ to z.

$$\sum_{n=1}^{\infty}\frac{f_n}{n^2}z^{-n} = \int_z^\infty z^{-1}\int_z^\infty p^{-1}F^*(p)dp$$

$$= \mathfrak{d}\left\{\frac{f(t)}{t^2}\right\}. \tag{16}$$

Continuing this we obtain (1, 2)

$$\mathfrak{d}\left\{\frac{f(t)}{t^k}\right\} = \underbrace{\int_z^\infty z^{-1}\int_z^\infty z^{-1}\int_z^\infty z^{-1}\cdots\int_z^\infty}_{k \text{ times}} p^{-1}F^*(p)dp. \tag{17}$$

For $k > 1$ the above equation is defined for $t > 0$; for $k = 1$, Eq. 10 is valid.

Although the above results appear to yield $z\left[\frac{f(t)}{t^k}\right]$ for $k > 1$ in a closed form, actually it is not so because of difficulty in evaluating integrals

in a closed form of the type $\int(\ln(a + bx)/x)$. Therefore it is not possible to obtain closed expressions for z transforms of functions like $f(t)/t^k$ for $k > 1$.[5] However, the usefulness of the result lies in evaluating infinite series since numerical evaluations of definite integrals of the above type are available.

3. z TRANSFORMS OF FUNCTIONS SATISFYING

$$\int_0^\alpha f_1(t, \beta)d\beta = \frac{f(t, \alpha)}{t}. \tag{18}$$

Theorem: If

$$\int_0^\alpha f_1(t, \beta)d\beta = \frac{f(t, \alpha)}{t}$$

then

$$\int_0^\alpha F_1^*(z, \beta)d\beta = \frac{1}{T}\int_z^\infty p^{-1}F^*(p, \alpha)dz + \lim_{n\to 0}\frac{f_n}{n}. \tag{19}$$

Note that $F_1^*(z, \beta)$, $F^*(z, \alpha)$ are just z transforms of $f_1(t, \beta)$ and $f(t, \alpha)$ with α, β as parameters and are not to be confused with modified z transforms (2).

Assumptions:

1. $f_1(t, \alpha) = 0$ for $t < 0$ which assures that $F_1^*(z, \beta)$ can be expanded in powers of z^{-1} only.

2. $f(t, \alpha) = 0$ as $t \to 0$ and $\lim_{n\to 0}\frac{f_n}{n}$ exists.

The importance of the assumptions becomes clear during the course of the proof.

Proof:

$$\int_0^\alpha f_1(t, \beta)d\beta = \frac{f(t, \alpha)}{t}.$$

Taking the z transform on both sides (6)

$$\sum_{n=0}^\infty\left[\int_0^\alpha f_1(t, \beta)d\beta\right]z^{-n} = \sum_{n=0}^\infty\frac{f(t, \alpha)}{t}z^{-n}. \tag{20}$$

Interchanging summation and integration with respect to α in Eq. 20 (provided the integral exists)

$$\int_0^\alpha\left[\sum_{n=0}^\infty f_1(nT, \beta)z^{-n}\right]d\beta = \sum_{n=0}^\infty\frac{f(nT, \alpha)}{nT}z^{-n}. \tag{21}$$

[5] For k negative integer, the z transform is easily obtainable as explained in (2, 5).

The term inside brackets on the left hand side of Eq. 21 is the z transform of $f_1(t, \beta)$

$$\int_0^a F_1^*(z, \beta)d\beta = \lim_{n\to 0}\frac{f(nT, \alpha)}{nT} + \frac{f(T, \alpha)}{T}z^{-1} + \frac{f(2T, \alpha)}{2T}z^{-2} + \cdots\frac{f(nT, \alpha)}{nT}z^{-n} + \cdots. \quad (22)$$

We recognize that

$$\frac{z^{-n}}{n} = \int_z^\infty p^{-(n+1)}dp. \quad (23)$$

The right hand side of Eq. 22 reduces to

$$\lim_{n\to 0}\frac{f(n, \alpha)}{n} + \frac{1}{T}\left[\int_z^\infty f(T, \alpha)p^{-2}dp + \int_z^\infty f(2T, \alpha)p^{-3}dp + \cdots\int_z^\infty f(nT, \alpha)p^{-(n+1)}dp\right] \quad (24)$$

which is

$$\lim_{n\to 0}\frac{f(n, \alpha)}{n} + \frac{1}{T}\int_z^\infty p^{-1}[F^*(p, \alpha) - f(0, \alpha)]dp \quad (25)$$

$f(0, \alpha) = 0$ by assumption. Hence,

$$\int_0^a F_1^*(z, \beta)d\beta = \lim_{n\to 0}\frac{f(n, \alpha)}{n} + \frac{1}{T}\int_z^\infty p^{-1}F^*(p, \alpha)dp. \quad (26)$$

Example:

Consider the z transform of $(1 - \cos \alpha t)/t$. Now

$$\int_0^a \sin \beta t d\beta = -\frac{\cos \beta t}{t}\Big|_0^a = \frac{-\cos \alpha t + 1}{t} = \frac{1 - \cos \alpha t}{t}. \quad (27)$$

Therefore

$$\mathfrak{z}\frac{1 - \cos \alpha t}{t} = \int_z^\infty p^{-1}F^*(p, \alpha)dp + \lim_{n\to 0}\frac{f_n}{n} \quad (28)$$

where

$$F^*(z, \alpha) = \mathfrak{z}(1 - \cos \alpha t) = \frac{(1 - \cos \alpha)z(z + 1)}{(z - 1)(z^2 - 2z \cos \alpha + 1)}. \quad (29)$$

Instead of evaluating the integral with respect to z, we can evaluate $\int_0^\alpha F_1^*(z, \beta)d\beta$, where

$$F_1^*(z, \beta) = \frac{z \sin \beta}{z^2 - 2z \cos \beta + 1}. \tag{30}$$

Since $\lim_{n\to 0} \frac{f_n}{n} = 0$,

$$\int_0^\alpha \frac{z \sin \beta}{(z^2 - 2z \cos \beta + 1)} d\beta = \int_z^\infty \frac{(1 - \cos \alpha)p(p + 1)}{(p - 1)(p^2 - 2p \cos \alpha + 1)} dp. \tag{31}$$

By actual computation these can be proved to be equal and therefore

$$\mathfrak{z} \frac{1 - \cos \alpha t}{t} = \tfrac{1}{2} \ln \frac{z^2 - 2z \cos \alpha + 1}{z^2 - 2z + 1}. \tag{32}$$

It is seen that, in this case, integration with respect to α is slightly easier than integration with respect to z.

Now

$$\sum_{n=1}^{\infty} \frac{1 - \cos \alpha n}{n} z^{-n} = \tfrac{1}{2} \ln \frac{z^2 - 2z \cos \alpha + 1}{z^2 - 2z + 1}. \tag{33}$$

By manipulating this expression and substituting appropriate values of z and α we can sum up a number of infinite series. First putting $z = -1$ we have

$$\sum_{n=1}^{\infty} \frac{1 - \cos \alpha n}{n} (-1)^n = \ln \cos \frac{\alpha}{2}. \tag{33a}$$

Now integration with respect to α is definitely quicker to evaluate infinite series since, as we see below, substitution of $z = (-1)$ in the integrand of the left hand side of Eq. 31 gives the summation very quickly.

$$\int_0^\alpha \frac{-\sin \beta}{2(1 + \cos \beta)} d\beta = \frac{-1}{2} \int_0^\alpha \tan \frac{\beta}{2} d\beta = \ln \cos \beta/2 \Big|_0^\alpha$$
$$= \ln \cos \alpha/2. \tag{33b}$$

This avoids the long process in evaluating the *definite* integral

$$\int_{-1}^{\infty} \frac{(1 - \cos \alpha)p(p + 1)}{(p - 1)(p^2 - 2p \cos \alpha + 1)} dp.$$

Let us derive a few well-known series using the result

$$\sum_{n=1}^{\infty} \frac{1 - \cos \alpha n}{n} z^{-n} = \tfrac{1}{2} \ln \frac{z^2 - 2z \cos \alpha + 1}{z^2 - 2z + 1}. \tag{34}$$

If $z = 1$ then the series diverges on the right hand side. Putting $z = (-1)$ in Eq. 34,

$$\sum_{n=1}^{\infty} \frac{1 - \cos \alpha n}{n} (-1)^n = \ln \cos \frac{\alpha}{2}. \tag{34a}$$

When $\alpha = \pi/2$,

$$\sum_{n=1}^{\infty} \frac{1 - \cos \frac{\pi}{2} n}{n} (-1)^n = \ln \frac{1}{\sqrt{2}} = -\tfrac{1}{2} \ln 2$$

that is,

$$-1+1-\frac{1}{3}-\frac{1}{5}+\frac{1}{3}-\frac{1}{7}-\frac{1}{9}+\frac{1}{5}-\frac{1}{11}-\frac{1}{13}+\frac{1}{7}\cdots = -\frac{1}{2} \ln 2. \tag{35}$$

When $\alpha = \pi$

$$\sum_{n=1}^{\infty} \frac{1 - \cos n\pi}{n} (-1)^n = \ln \cos \frac{\pi}{2} = \infty,$$

that is,

$$-2(1 + \tfrac{1}{3} + \tfrac{1}{5} \cdots) = \infty. \tag{36}$$

Therefore the sum $\sum_{n=1}^{\infty} (1/(2n+1))$ does not converge. Hence, by giving any value of α we can get a closed form and it also tells us whether the series converges or diverges. Thus Eq. 26 furnishes us two ways to evaluate an integral providing (18) is satisfied and as the example above has shown, we can use the equivalence of the integrals to sum up certain infinite series very easily.

4. SUMMATION OF INFINITE SERIES

One of the problems in the study of infinite series is to obtain closed expressions and operational methods have been used to obtain such results (7, 8). z transform by its very definition lends itself very easily to such summations and there are extensive tables of z transforms for various discrete functions of time. In this section we will use the results of the previous sections to develop closed expressions for summing certain types of infinite series of the type $\sum_{n=0}^{\infty} (f_n/n^k)$.[6] Summation of

[6] The summation of the type $\sum_{n=0}^{\infty} (f(n+m)/n^k)$, $0 \leq m \leq 1$, can also be obtained by simple modification of Eq. 17 to read:

$$\sum_{n=0}^{\infty} \frac{f(n+m)}{n^k} = \lim_{z \to 1} \left[\underbrace{\int_z^{\infty} z^{-1} \int_z^{\infty} z^{-1} \cdots \int_z^{\infty}}_{k \text{ times}} p^{-1} F^*(p, m) dp \right]$$

where

$$F^*(p, m) = z\delta_m f(t)\,|_{z=p}$$
$$\delta_m f(t) = \text{modified (2) } z \text{ transform of } f(t).$$

series of the type $\sum_{n=0}^{\infty} f_n g_n$ using convolution z transform has been discussed in the literature (9, 10).

Example 1

Using convolution z transform we have developed

$$\mathfrak{z}\left[\frac{1}{t}\right] = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} = \ln \frac{z}{z-1} \qquad t > 0. \tag{37}$$

Substituting $z = -1$,

$$\sum_{n=1}^{\infty} \frac{1}{n} (-1)^n = \ln \frac{-1}{-1-1} = -\ln 2$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n} (-1)^{n+1} = \ln 2. \tag{38}$$

Example 2

Letting $z^{-1} = y$ in Eq. 37 and integrating with respect to y from $\int_0^y$, we obtain

$$\int_0^y \sum_{n=1}^{\infty} \frac{1}{n} y^n = \int_0^y \ln \frac{1}{1-y}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} y^{n+1} = (1-y) \ln (1-y) + y \tag{39}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} z^{-(n+1)} = (1-z^{-1}) \ln (1-z^{-1}) + z^{-1}.$$

With $z = 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1. \tag{40}$$

With $z = (-1)$,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} (-1)^{n+1} = 2 \ln 2 - 1. \tag{41}$$

Example 3

$$\sum_{n=1}^{\infty} \frac{1}{n} z^{-n} = \ln \frac{z}{z-1}.$$

Multiplying by z^{-1} and integrating between the limits $\int_z^{\infty}$ we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} z^{-n} = \int_z^{\infty} p^{-1} \ln \frac{p}{p-1} \, dp. \tag{42}$$

This integral is not immediately evaluable. However, changing the variable of integration on the right-hand side from $p^{-1} = x$ reduces the integral to the form $\int_0^{z^{-1}} x^{-1} \ln (1/(1-x))dx$. Tables are available for this integral between the limits $\int_0^1$. Substituting $z = 1$ and using tables, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{43}$$

Summation $\sum_{n=1}^{\infty} (1/n^k)$ for even values of $k > 2$ has been developed using other techniques (**4**, **7**, **8**).

Example 4

Evaluations of the following infinite series can be done easily by manipulation of Eq. 37.

$$\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2^{n+1}} = 8 \tag{44}$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \tag{45}$$

$$\sum_{n=0}^{\infty} \frac{n+1}{2^n} = 4. \tag{46}$$

Example 5

To evaluate $\sum_{n=0}^{\infty} \left[\frac{\sin \alpha n}{n}\right]$,

$$\sum_{n=0}^{\infty} \frac{\sin \alpha n}{n} z^{-n} = \mathfrak{z}\left[\frac{\sin \alpha t}{t}\right]. \tag{47}$$

Recognizing that

$$\frac{\sin \alpha t}{t} = \int_0^{\alpha} \cos \beta t d\beta, \tag{48}$$

we can evaluate the z transform in either of the two ways mentioned in Sec. 3. If we do that we get

$$\sum_{n=0}^{\infty} \frac{\sin \alpha n}{n} z^{-n} = \alpha + \frac{1}{2j} \ln \frac{z - e^{-j\alpha}}{z - e^{+j\alpha}}. \tag{49}$$

For $\alpha = \pi/2$ and $z = 1$,

$$\sum_{n=0}^{\infty} \frac{\sin \frac{\pi}{2} n}{n} = \frac{\pi}{2} + \frac{1}{2j} \ln e^{j(\pi/2)}$$

$$= \frac{3\pi}{4}. \tag{50}$$

Recognizing that

$$\lim_{n\to 0} \frac{\sin \frac{\pi}{2} n}{n} = \frac{\pi}{2}, \tag{51}$$

we get

$$\sum_{n=1}^{\infty} \frac{\sin \frac{\pi}{2} n}{n} = 1 - \tfrac{1}{3} + \tfrac{1}{5} \cdots = \frac{\pi}{4}. \tag{52}$$

This series known after Leibniz (11) is known for its clarity and simplicity and also affords a way of determining the value of π.

Example 6

The z transform for $(f(t)/(t + a))$ has been derived (2):

$$z \frac{f(t)}{t + a} = z^a \int_z^{\infty} p^{-(1+a)} F^*(p) dp, \tag{53}$$

where a is a positive constant > 0.

If $f(t)$ is a unit step function then $F^*(z) = \dfrac{z}{z - 1}$ and assuming a to be positive integer

$$\mathfrak{Z} \frac{1}{t + a} = \sum_{n=0}^{\infty} \frac{1}{n + a} z^{-n} = z^a \int_z^{\infty} p^{-(1+a)} \frac{p}{p - 1} dp \tag{54}$$

$$= z^a \int_{\infty}^{z} \left[\frac{1}{p^a} + \frac{1}{p^{a-1}} + \cdots \frac{1}{p} - \frac{1}{p - 1} \right] dp$$

$$= z^a \left[\ln \frac{z}{z - 1} - \left\{ \frac{z^{-(a-1)}}{a - 1} + \frac{z^{-(a-2)}}{a - 2} + \cdots + \frac{z^{-1}}{1} \right\} \right] \tag{55}$$

$$= z^a \left[\ln \frac{z}{z - 1} - \sum_{p=1}^{a-1} \frac{1}{p} z^{-p} \right].$$

This result can be easily verified by substituting the known series for $\ln \dfrac{z}{z - 1}$ and getting back the result $\sum_{n=0}^{\infty} (1/(n + a)) z^{-n}$.

Example 7

z transforms of $(1/(n+a)(n+b))$, with a, b positive integers, are

$$\sum_{n=0}^{\infty} \frac{1}{(n+a)(n+b)} z^{-n} = \sum_{n=0}^{\infty} \frac{z^{-n}}{b-a}\left(\frac{1}{n+a} - \frac{1}{n+b}\right) \tag{56}$$

$$= \frac{1}{b-a}\left[z^a\left\{\ln\frac{z}{z-1} - \sum_{p=1}^{a-1}\frac{1}{p}z^{-p}\right\} - z^b\left\{\ln\frac{z}{z-1} - \sum_{p=1}^{b-1}\frac{1}{p}z^{-p}\right\}\right] \tag{57}$$

$$= \frac{1}{b-a}\left[\left(\ln\frac{z}{z-1}\right)(z^a - z^b) - z^a\sum_{p=1}^{a-1}\frac{1}{p}z^{-p} + z^b\sum_{p=1}^{b-1}\frac{1}{p}z^{-p}\right].$$

As an example let $a = 1$, $b = 2$:

$$\mathfrak{z}\frac{1}{(n+1)(n+2)} = \left(\ln\frac{z}{z-1}\right)(z - z^2) + z^2 z^{-1}$$

$$= z + (z - z^2)\ln\frac{z}{z-1}$$

$$= z\left[1 + (1-z)\ln\frac{z}{z-1}\right]. \tag{58}$$

Similarly we can develop $\mathfrak{z}$ transforms for $(1/(n+a)(n+b)(n+c)\cdots)$. The above results are in agreement with those of Stone (12).

If we substitute $z = 1$ we have from (58)

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1. \tag{59}$$

With $z = -1$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)} = 2\ln 2 - 1. \tag{60}$$

The above examples have shown that summation of infinite series is equivalent to evaluating a certain integral and by obtaining closed forms for expressions of the type $\sum_{n=0}^{\infty}(f_n/n^k)z^{-n}$ we are able to tell whether a particular series is convergent or divergent.

CONCLUSIONS

In this paper have been shown: (*a*) the equivalence of two forms of integrals, which could be used in evaluating an infinite series in a simple manner; (*b*) the problems associated with and care required in integrating functions involving branch point in the z plane. The integra-

tion is useful in obtaining the z transform of functions with an irrational power of z. Further investigation into evaluating such functions is indeed warranted. Finally, (*c*) a closed form of infinite series using the convolution integral is obtained and a few examples are discussed. Extension of this method to obtain closed forms for certain other series is possible and conditions on convergence or the possibility of obtaining a closed form of the series is readily assessed from the form of the integrand in the convolution integral. Further extension of the convolution for obtaining a closed form of a finite series of the form

$$\sum_{n=0}^{k} \frac{f_{n+m}}{n^k} \qquad 0 \leq m \leq 1$$

is possible and worthy of further investigation.

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DESIGN RELATIONS FOR RESONANT POST WAVEGUIDE FILTERS

BY

ROBERT D. WANSELOW [1]

SUMMARY

This paper presents a simple design procedure for resonant post waveguide filters. Such microwave circuits commonly are physically realized as ladder networks consisting of cascaded single post resonant cavities or double post resonant elements coupled by quarter-wave sections of transmission line. With this simplified technique, design effort is minimized by the use of pertinent parameter relationships expressed in the form of a few graphs. A comparison is made of the theoretical and measured characteristics of two experimental filters each employing one of the two post configurations. The results show that this design procedure is quite adequate for most engineering applications.

INTRODUCTION

This paper considers waveguide filters that are of the band-pass type. These filters employ inductive posts as the primary reactive element in either of two basic configurations. The first and most common configuration is the cavity type where two identical single posts (longitudinally spaced approximately a half wavelength along the guide) make up one stage of the filter. The other configuration is the resonant element type where one filter section consists of a pair of identical posts (double posts) positioned transversely in the guide. This configuration is less known, but exhibits many desirable characteristics. Multistage filters are considered where both of these primary post configurations are employed as cascaded resonators coupled by quarter wavelengths of transmission line as shown in Figs. 1*a* and 1*b*. The theory of quarter-wavelength coupling has been discussed by many (1, 2, 3)[2] quite adequately. As the Q of a given microwave filter is increased, the fabrication and adjustment of directly coupled resonant structures become increasingly more difficult. Therefore, by employing quarter-wave coupling between the filter resonators, the mechanical tolerances on the dimensions of the coupled elements are eased with no noticeable degradation of filter performance. This is especially true where single post cavity-type filters are required with loaded Q's ranging from 15 to 200. Since the double post resonant structure is inherently a low Q type circuit, the quarter-wave coupling of this type of resonator produces a filter which is reduced in length by a factor of two over a direct coupled cavity filter and by a factor of approximately three over a

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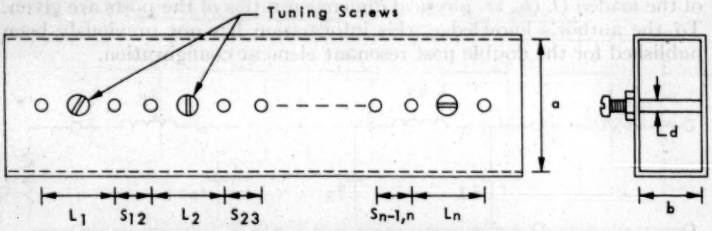
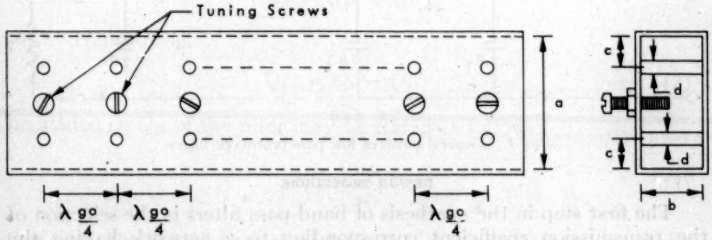
[2] The boldface numbers in parentheses refer to the references appended to this paper.

quarter-wave coupled cavity-type filter, each possessing the same number of sections. The double post configuration is a very applicable type resonant circuit for filters whose loaded Q's vary between 4 and 20. It should be noted that the correlation between theory and practice may be in error by as much as 5 per cent when either of these basic post configurations is employed when requiring loaded Q's near their physically realizable limits.



a, n - Section Quarter-Wave Coupled
Single Post Resonant Cavity Filter



b, n - Section Quarter-Wave Coupled
Double Post Resonant Element Filter

FIG. 1. Two principal types of quarter-wave coupled resonant post waveguide filters.

The tuning of a single post resonant cavity is accomplished by one tuning screw symmetrically placed between the two identical posts of the cavity, and with a double post resonant element a screw is also placed between these two transversely positioned posts. Mumford (4) and Reed (5) have discussed to some extent filters employing single post resonant cavities and Alstadter and Houseman (6) have suggested the use of double post filters. The theory of the double post as a waveguide obstacle was developed by Gruenberg (7). Thus this paper

coordinates some of this information previously published together with some additional experimental design data which have proven very valuable for the rapid design of waveguide filters employing posts as the basic reactive waveguide obstacle.

The simple design procedure presented here details the development of quarter-wave coupled waveguide filters employing either of the two basic post configurations described. Very useful experimental curves of the loaded Q, Q_L, *vs.* physical dimension ratios of the posts are given. To the author's knowledge, this information has not previously been published for the double post resonant element configuration.

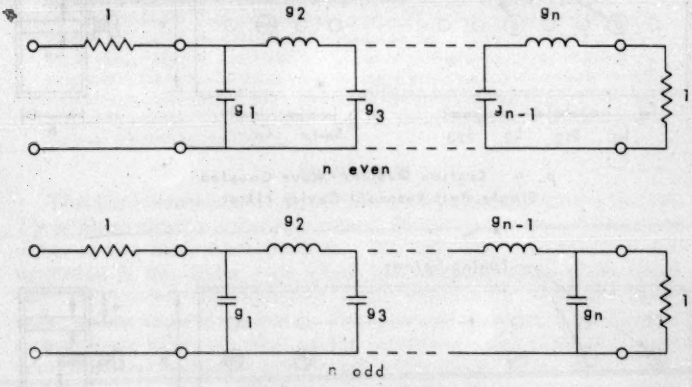


FIG. 2. General forms of low pass prototype filters.

DESIGN PROCEDURE

The first step in the synthesis of band-pass filters is the selection of the transmission coefficient corresponding to a network having the desired rejection-band characteristics. Two of the commonly used transfer functions are the Butterworth or maximally flat and the Tschebyscheff distributions. For given band-rejection characteristics or skirt selectivity, fewer resonant elements or stages are required when the synthesis is based on a Tschebyscheff distribution of the first kind with small tolerable ripple rather than a Butterworth distribution. Thus, upon selection of an appropriate transmission coefficient, the corresponding low pass prototype n-section filter, as shown in Fig. 2, can be synthesized from which a low-pass to band-pass transformation is to be made. From the low-pass prototype the normalized reactive parameters, g_k, can be obtained for use in the following design procedure. The transformation from a low-pass ladder network to its transmission-line quarter-wave coupled equivalent circuit of a band-pass filter is

shown schematically in Fig. 3. The loaded Q, Q_{Lk}, of each resonator will be defined in the formulation of the following design equations.

The effective Q, Q_k, of the k^{th} resonator of either a quarter-wave coupled cavity or resonant element type band-pass filter is equal to the loaded Q, Q_{Lk}, of the k^{th} resonator, plus the Q due to the quarter-wave

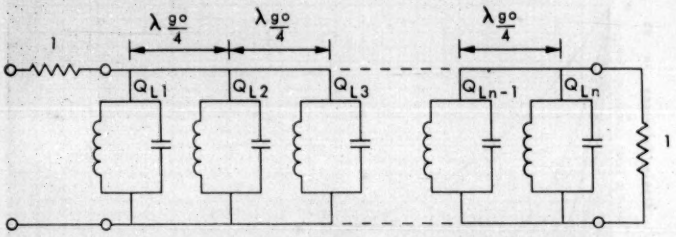


FIG. 3. Quarter-wave coupled equivalent circuit of a band-pass filter.

lines, Q_λ, which are connected to this resonator. Thus, we have the following relation:

$$Q_k = Q_{Lk} + Q_\lambda \tag{1}$$

and, by transposing,

$$Q_{Lk} = Q_k - Q_\lambda. \tag{2}$$

The loaded Q, Q_0, of the filter may be stated as follows:

$$Q_0 = \frac{\omega_0}{\omega_1'} \tag{3}$$

where ω_0 is the center frequency and ω_1' is the 3-db bandwidth of the band-pass filter. The effective Q, Q_k, of the k^{th} resonator may be equated as follows:

$$Q_k = \frac{g_k \omega_0}{2\omega'} \tag{4}$$

where g_k is the normalized reactive low pass prototype parameter and ω' is the ripple tolerance bandwidth for Tschebyscheff designs and the 3-db bandwidth for maximally flat or Butterworth distributions. From Cohn (8) a relation between normalized values of ω_1' and ω' is shown in Eq. 5,

$$3 \text{ db} = 10 \log_{10} [1 + (10^{A_m/10} - 1) \cosh^2 (n \cosh^{-1} \alpha)] \tag{5}$$

where A_m is the ripple tolerance level in db corresponding to the normalized $\omega' = 1$ point, n is the number of resonators in the filter, and $\alpha = \frac{\omega_1'}{\omega'}$. Some bandwidth correction curves for converting ripple tolerance bandwidth to 3 db bandwidth are shown in Fig. 4 from which

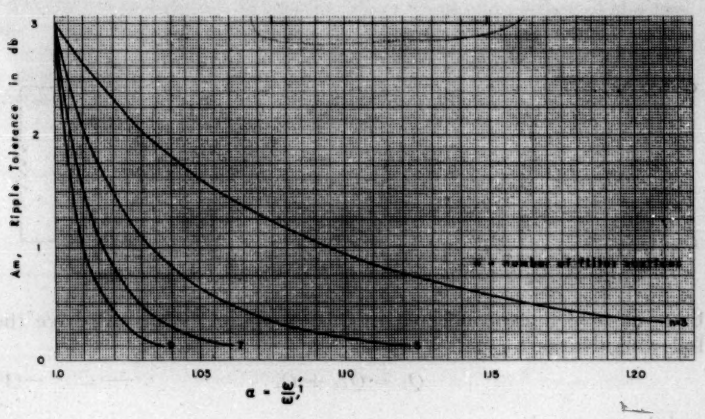


FIG. 4. Bandwidth correction curves.

α (the bandwidth correction factor) may be obtained. An approximate form of the loading upon a given resonator by its respective quarter-wave connecting line is

$$Q_\lambda = m\left(\frac{\lambda_{g0}}{\lambda_0}\right)^2\frac{\pi}{8} \tag{6}$$

where m is the integer 1 or 2 depending upon whether the resonator is connected to one or two quarter-wave lines, respectively, and λ_{g0} is the guide wavelength and λ_0 the free space wavelength corresponding to the filter center frequency. Therefore, by combining Eqs. 3, 4 and 6, Eq. 2 may be summarized in a more useful form with equal input and output impedances

$$Q_{Lk} = \alpha\frac{g_k}{2}Q_0 - m\left(\frac{\lambda_{g0}}{\lambda_0}\right)^2\frac{\pi}{8} \tag{7}$$

where

$$k = 1, 2, \ldots, n$$

$$m = \begin{cases} 1 \text{ for } k = 1 \text{ and } n \\ 2 \text{ for } k = 2, 3, \ldots, n-1. \end{cases}$$

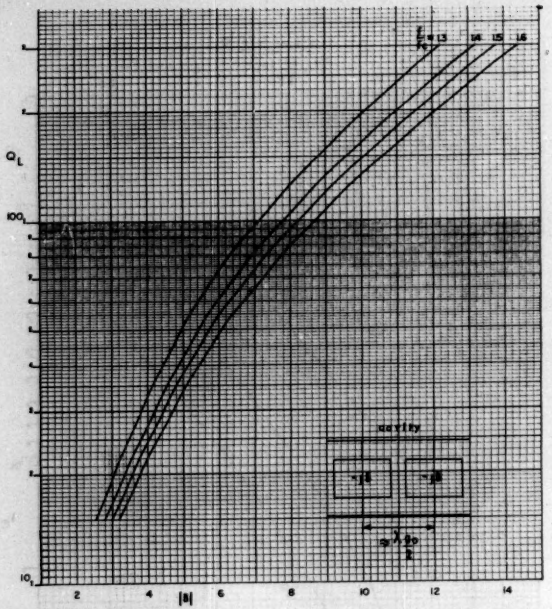


FIG. 5. The relation between the loaded Q, Q_L, and the normalized susceptance, B, of a resonant cavity formed by two shunt inductive susceptances.

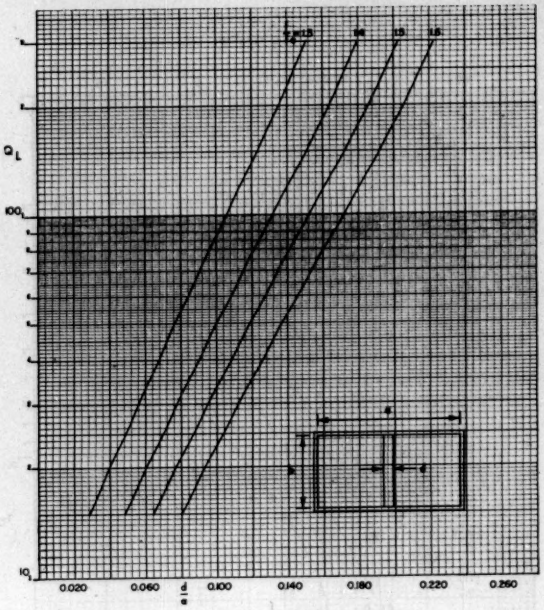
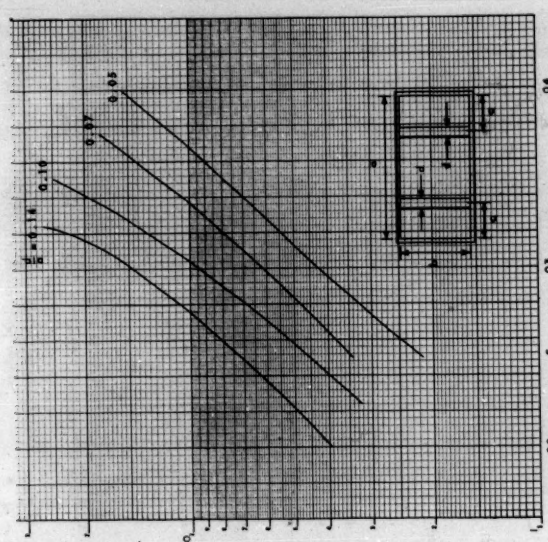
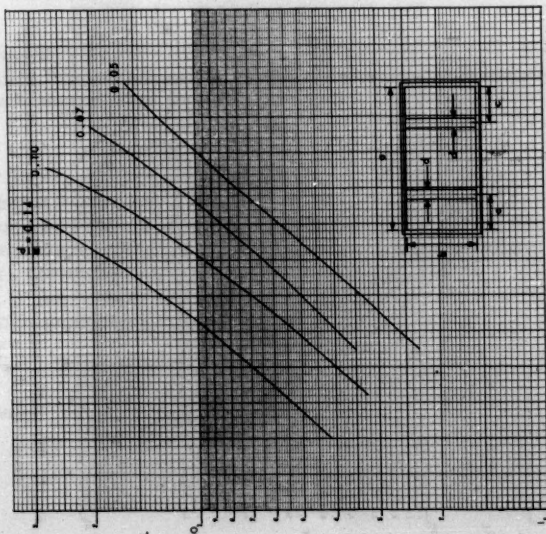


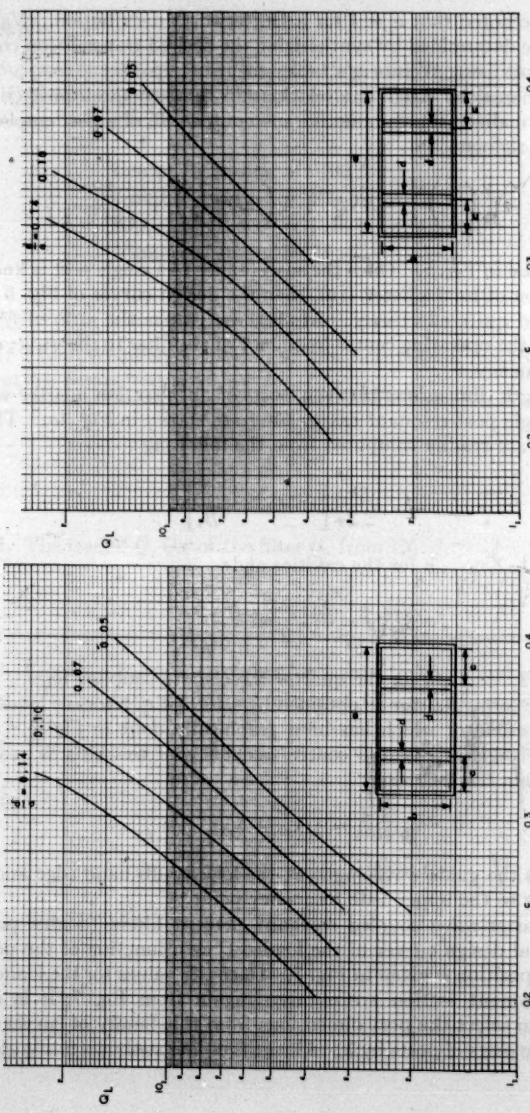
FIG. 6. Design curves for the loaded Q, Q_L, *vs.* the d/a ratio for single post resonant cavities.



b. $f/f_e = 1.4$



a. $f/f_e = 1.3$



c. $f/f_c = 1.5$ *d.* $f/f_c = 1.6$

FIG. 7. Design curves for the loaded Q. Q_L *vs.* the c/a ratio for double post resonant elements.

It should be noted that $\alpha = 1$ for a Butterworth distribution. Values of g_k may be determined by synthesizing the desired transmission coefficient or by utilizing Weinberg's tables (9).

As developed by Reed, the relationship between the loaded Q, Q_L, and the normalized shunt inductive susceptance, B, of a post employed as a cavity element is

$$Q_L = \frac{1}{4}\left(\frac{\lambda_g}{\lambda}\right)^2\left[-B\sqrt{B^2+4}\left(\tan^{-1}\frac{2}{B}\right)+\frac{2B^2}{\sqrt{B^2+4}}\right] \tag{8}$$

and is plotted in Fig. 5. Once the susceptance of a single post is known as a function of its diameter (10), then the design curves of Fig. 6 can be employed quite effectively. In all these graphs the ratio of f/f_c is the ratio of the operating center frequency of the filter to the waveguide cut-off frequency.

The length dimensions between posts of an n-section quarter-wave coupled single post resonant cavity filter are shown in Fig. 1*a*. These dimensions are readily computed from the following:

$$L_k = \frac{\lambda_{g0}}{2\pi}\left[\pi - \tan^{-1}\frac{2}{B_k}\right] \tag{9}$$

where $k = 1, 2, \ldots, n$ for the cavities and

$$S_{k-1,k} = \frac{L_{k-1}+L_k}{2} - \frac{\lambda_{g0}}{4} \tag{10}$$

where $k = 2, 3, \ldots, n$ for the quarter-wavelength coupling lines. The susceptance term, B_k, can be obtained by solving Eq. 8 for the loaded Q of the k^{th} resonator which has been graphically done in Fig. 5. The guide wavelength λ_{g0} at the center frequency of the filter is readily obtained from the following:

$$\lambda_{g0} = \tfrac{1}{2}(\lambda_{g1} + \lambda_{g2}) \tag{11}$$

where λ_{g1} is the guide wavelength at the lower 3-db band edge and λ_{g2} corresponds to the upper 3-db frequency point.

For quarter-wave coupled resonant element filters (double posts) experimental design curves of the loaded Q as a function of the posts' transverse position is given in Fig. 7. Once the values for the loaded Q of each resonant section are obtained again from Eq. 7, it is only necessary to select the post dimensions from this family of curves and to space the pairs of posts by $\lambda_{g0}/4$ lengths of line as shown in Fig. 1*b*.

Example

Suppose one is required to design a three-stage quarter-wave coupled single post resonant cavity-type filter in X-band waveguide with the following basic specifications:

Center frequency = 9100 mc
3-db bandwidth = 100 mc
Ripple tolerance = 1 db is permissible
Number of stages = 3

Procedure

1. From Weinberg's tables (9) the normalized reactive low-pass prototype parameters are:

$$g_1 = 2.0236$$
$$g_2 = 0.9941$$
$$g_3 = g_1 = 2.0236$$

2. The loaded Q, Q_0, of the filter is, from Eq. 3,

$$Q_0 = \frac{2\pi}{2\pi} \times \frac{9100}{100} = 91$$

3. From the bandwidth correction curves in Fig. 4, a 3-section filter ($n = 3$) with a 1-db ripple tolerance level ($A_m = 1$ db) exhibits a value α of 1.094.

4. The center guide wavelength, λ_{g0}, can be obtained from Eq. 11

where

$$\lambda_{g1} = 1.892 \text{ in. for } f_1 = 9050 \text{ mc}$$
$$\lambda_{g2} = 1.849 \text{ in. for } f_2 = 9150 \text{ mc}$$

in standard RG-52/U guide. Thus:

$$\lambda_{g0} = \tfrac{1}{2}(1.892 + 1.849)$$
$$= 1.870 \text{ in.}$$

which corresponds to a free space wavelength, λ_0, of 1.296 in.

5. From Eq. 7 the loaded Q, Q_{Lk}, of each resonator can now be calculated.

$$Q_{L1} = 1.094 \left(\frac{2.0236}{2}\right) 91 - (1) \left(\frac{1.870}{1.296}\right)^2 \frac{\pi}{8}$$

$$= 99.7$$

$$Q_{L2} = 1.094 \left(\frac{0.9941}{2}\right) 91 - (2) \left(\frac{1.870}{1.296}\right)^2 \frac{\pi}{8}$$

$$= 47.8$$

$$Q_{L3} = Q_{L1} = 99.7$$

6. The ratio of the center frequency to the waveguide cut-off frequency is

$$\frac{f}{f_c} = \frac{9100}{6556} = 1.39$$

7. Thus from the curves of Fig. 5, using $\frac{f}{f_c} = 1.4$, the normalized inductive susceptance can be obtained.

$$B_1 = 7.8$$

$$B_2 = 5.3$$

$$B_3 = B_1 = 7.8$$

8. From the curves of Fig. 6 the $\frac{d}{a}$ ratio of each post can be obtained

$$\frac{d_1}{a} = 0.132$$

$$\frac{d_2}{a} = 0.098$$

$$\frac{d_3}{a} = \frac{d_1}{a} = 0.132$$

Therefore, since $a = 0.900$ in.

$$d_1 = d_3 = 0.119 \text{ in.}$$

$$d_2 = 0.088 \text{ in.}$$

9. From information obtained in steps 4 and 7, Eqs. 9 and 10 can now be employed to determine the space separation of the posts:

$$L_1 = L_3 = \frac{1.870}{2 \times \pi}\left[\pi - \tan^{-1}\frac{2}{7.8}\right]$$

$$= 0.860 \text{ in.}$$

$$L_2 = \frac{1.870}{2 \times \pi}\left[\pi - \tan^{-1}\frac{2}{5.3}\right]$$

$$= 0.830 \text{ in.}$$

$$S_{1,2} = S_{2,3} = \frac{0.860 + 0.830}{2} - \frac{1.870}{4}$$

$$= 0.378 \text{ in.}$$

The dimensioning of an n-section filter of this type is shown in Fig. 1*a*.

EXPERIMENTAL RESULTS

Figure 8 shows the measured and theoretical insertion loss response characteristics of a 3-section quarter-wave coupled single post resonant

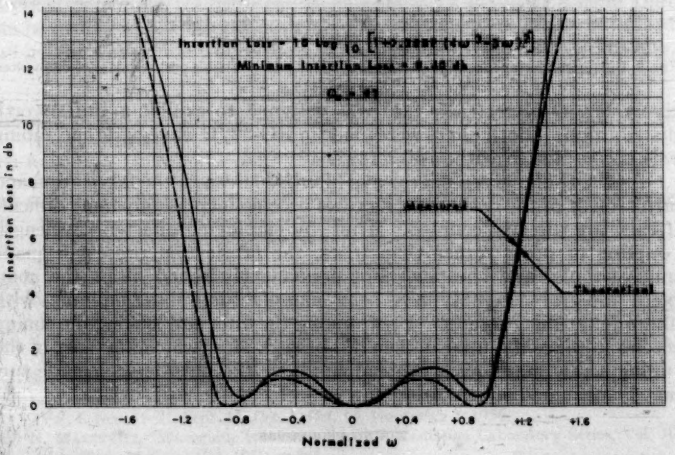


FIG. 8. Tschebyscheff response characteristics of a three section single post resonant cavity filter.

cavity-type waveguide filter where the pass-band ripple tolerance was designed to be 1 db. These curves exhibit a close correlation between design and theory; the step-by-step design for this filter is given in the

preceding section of this paper. A comparison between the experimental and theoretical results of a 4-section quarter-wave coupled double post resonant element type filter is shown in Fig. 9. This rela-

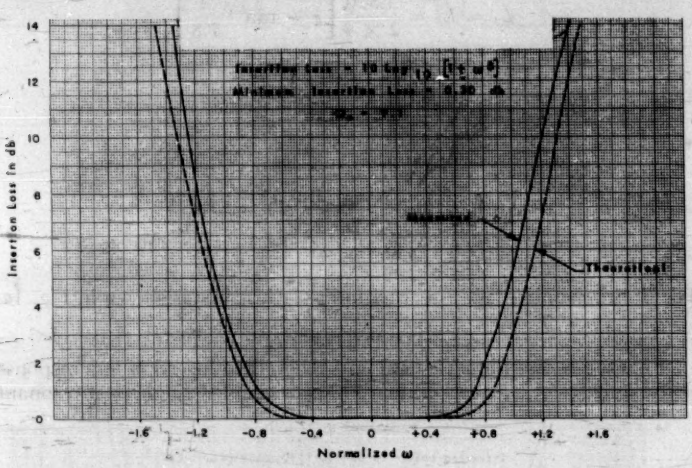


FIG. 9. Butterworth response characteristics of a four section double post resonant element filter.

tively low Q filter ($Q_0 = 9.1$) was designed to exhibit a Butterworth distribution. Both filters were made in RG-52/U waveguide and tuning was accomplished with 4–40 screws.

As the Q of a given microwave filter is increased, the fabrication and adjustment of the resonant structures become increasingly more difficult if they are directly coupled. Thus, by employing quarter-wavelength coupling of posts rather than irises, the fabrication process is made much simpler and cheaper. For permanently tuned filters the complete integral assembly has been manufactured in aluminum with aluminum tuning screws. Upon tuning the filter and locking the tuning screws in place the filter is then dip-brazed, after which any part of the tuning screw mechanism protruding above the outside waveguide surface is machined off, resulting in a smooth outside wall surface.

TUNING PROCEDURE

The tuning procedure for quarter-wave coupled filters involves a method of quarter-wave shifting the VSWR nulls on the input transmission line to the filter. A standing wave detector is inserted between a matched generator and the input side of the filter with a matched detector mount on the terminating side of the filter. Initially, all the

resonant elements are detuned by inserting the tuning screws across the guide with the generator adjusted to the desired resonant center frequency. The positions P and P' of two minima of the voltage standing wave are determined. The first cavity or resonant element is then tuned with the first tuning screw until one of the minima coincides with a point R, halfway between P and P', that is, until the standing wave minima are shifted by $\lambda_{g0}/4$. The second resonant circuit is now tuned with the second screw until the minima shift back to their original points, P and P'. Thus, successive resonators are tuned by alternately adjusting the position of a standing wave minimum to occur at points P and R. The last resonator is adjusted so that maximum power will flow to the terminating detector. Upon completion of this tuning procedure, the filter should be checked at the edges of the pass band to make sure that the over-all behavior of the filter is satisfactory.

CONCLUSION

Use of the procedures and graphs presented enables the microwave engineer easily to design waveguide filters with loaded Q's ranging between 4 and 200. This procedure is sufficiently adequate for most engineering applications. Graphs of the loaded Q's as a function of the physical dimension ratios of the posts have been given in a form that can be readily applied to the design of a waveguide filter in any standard full height waveguide where the a/b ratio is approximately two to one.

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THEORY OF NONLINEAR CONTROL *

BY

Y. H. KU [1]

ABSTRACT

In the introduction the author discusses the nonlinearity in Nature as exemplified by a simple pendulum, inverse square laws of gravitation and electromagnetism, electronics, and planetary motion. Nonlinear feedback control systems have been analyzed in symposia sponsored by Polytechnic Institute of Brooklyn, American Society of Mechanical Engineers, and Institute of Radio Engineers. Even nonlinear system synthesis has been attempted.

Part II discusses the three stages in the study of physical systems: (1) Development of linear system analysis and idealization of actual systems as linear; (2) Recognition of nonlinear systems and analysis of such systems as they actually are; and (3) Introduction of the new concept of nonlinear control and improvement of the performance of linear and nonlinear systems. Under (3) it is recognized that besides linear compensation of systems with nonlinearities the logical alternative might be the nonlinear feedback control of physical systems for either stability or optimum response.

Part III starts with a review of linearizing techniques developed by Kochenburger, Johnson, Klotter, and others, known as the describing function method. Other quasi-linearization, equivalent linearization, and piecewise linearization techniques by Chen, Booton, Bass, and Stern are discussed.

Part IV deals with the phase-space, the switching surface, and related concepts. The method of simultaneous phase-plane equations in a multi-degree-of-freedom system is especially adaptable to nonlinear feedback control system analysis. The phase-space approach has been used by Kalman, Hopkin, and others for the design of nonlinear servomechanisms.

Part V covers new analytic techniques and new transforms. Papers by Pipes, Madwed, L. J. Lewis, Gillies, Stout, E. Weber, and others are reviewed. Taylor-Cauchy transforms for nonlinear systems are presented.

Part VI discusses theory of nonlinear systems, including recent developments in stability theory and criteria, optimum nonlinear control systems, sampled-data systems, and nonlinear systems with random inputs. These recent developments give substantial support to the author's statement: "The newest aspect of feedback control is the development of the theory of nonlinear control."

I. INTRODUCTION

1. *Nature Is Nonlinear*

As nature embodies a manifold of beautiful phenomena, it cannot be limited by a linear relationship between elementary quantities. Thus a

* Presented at the First International Congress of IFAC on Automatic Control, Moscow, U.S.S.R., June 27 to July 6, 1960. The complete English version appears here for the first time. While a Russian translation of the complete paper is available at the Moscow Congress, an abridged English version is to appear in the IFAC Congress Proceedings, published by Butterworths Publications, Ltd., London.

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simple pendulum as discovered by Galileo represents a simple harmonic motion, and yet the equation of a pendulum

$$\theta'' + (g/l) \sin \theta = 0 \tag{1}$$

is a second-order *nonlinear* differential equation, in which θ denotes the angular displacement, θ'' denotes $d^2\theta/dt^2$, the angular acceleration, g denotes a constant due to gravity, l denotes length, and t denotes time.

Kepler's laws of planetary motion can be expressed as follows;

$$M_1[r'' - r(\theta')^2] + \frac{\gamma M_1 M}{r^2} = 0 \tag{2}$$

$$M_1[2r'\theta' + r\theta''] = 0 \tag{3}$$

where M_1 denotes the mass of a planet, M denotes the mass of Earth, γ denotes the gravitational constant, r denotes the distance between the planet and Earth, and θ denotes the angle. Both Eqs. 2 and 3 are nonlinear. It is of interest to note that not only Newton's law of gravitation is an inverse square law, but also Coulomb's law of force between electric charges and the similar law of force between magnetic poles are inverse square laws.

Van der Pol's equation

$$x'' - \epsilon(1 - x^2)x' + x = 0 \tag{4}$$

representing an electronic oscillator, in which x denotes a normalized value of grid voltage (ratio of grid voltage to saturation voltage), is nonlinear and gives rise to *limit cycles*. Transistors and magnetic amplifiers are both nonlinear devices.

It has been remarked that the tremendous progress in the physical sciences is in large part due to the fact that most natural phenomena are at least amendable to a linear approximation. So long as one carefully notes the natural limitations of a *linear* spring (for which Hooke's law holds) or the linear portion of a B–H curve (before saturation), simple relations can be utilized to study natural phenomena. However, one is at the same time aware of the complexity of nature and wishes to study nature as it really is.

2. *Feedback Control Systems*

It was only twenty-five years ago that H. L. Hazen published his paper entitled "Theory of Servomechanisms" (**1**).[2] At Cranfield Conference 1951, G. S. Brown discussed "Feedback System Engineering—A Challenging Educational Objective" (**2**). Control systems are, in general, classified as open-loop (open-cycle) systems (Fig. 1) or as

[2] The boldface numbers in parentheses refer to the references appended to this paper,

closed-loop (closed-cycle) systems (Fig. 2) depending upon whether a comparison of the controlled quantity with the reference or desired value is absent or present. Thus a system of time-operated traffic lights represents an open-loop regulator and a thermostat temperature control system represents a closed-loop regulator. According to Brown and Campbell (3), a servomechanism is defined as "an error-sensitive, follow-up, amplifying system permitting a wide range of input command remotely located from the element being controlled."

Truxal (4) pointed out: "The situations of primary interest to the designer of feedback systems are those in which he is using feedback for a definite purpose." This definite purpose falls into one of the five categories;

1. Simplification of the practical realization of a specified transfer function.
2. Reduction of the sensitivity of the system characteristics to changes in parameters.
3. Control over the impedance seen looking into the system at various points.
4. Simultaneous control over a number of different transmittances, corresponding to multiple inputs to the single system.
5. Control over the stability, or instability, of the system, as in an oscillator circuit.

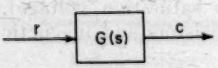

FIG. 1. Block diagram of an open-loop system.

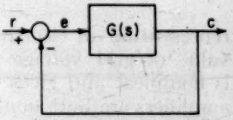

FIG. 2. Block diagram of a closed-loop system.

In the first application, feedback is used to overcome limitations of the physical components. In the other applications, feedback is introduced to effect a specific change in the characteristics of the system. The above remarks apply to both linear and nonlinear feedback control systems.

N. Wiener (5) has borrowed the term *cybernetics* from Ampère to mean the science of "control and communication in the animal and the machine." Thus a cybernetic system may refer to a system which is designed for the communication of information or to a system which is designed for the control of one or more variables. H. S. Tsien (6) remarked in his Preface to *Engineering Cybernetics:* "A distinguishing feature of this new science is the total absence of considerations of energy, heat, and efficiency, which are so important in other natural sciences. In fact, the primary concern of cybernetics is on the qualita-

tive aspects of the interrelations among the various components of a system and the synthetic behavior of the complete mechanism." Wiener's recent book (7) has examined the role of nonlinear process in physics, mathematics, electrical engineering, physiology, and communication theory.

The emphasis on nonlinear circuit analysis in automatic control systems in recent years can be seen from the following list of symposia:

1. Conference on Automatic Control, Cranfield, 1951.
2. Symposia on Nonlinear Circuit Analysis 1953 and 1956 (Polytechnic Institute of Brooklyn, New York).
3. Symposium on Nonlinear Control Systems 1956 (American Society of Mechanical Engineers).
4. Fachtagung Regelungstechnik Heidelberg 1956.
5. Symposium on Nonlinear Control 1958 (Institute of Radio Engineers).

II. THREE STAGES IN THE STUDY OF PHYSICAL SYSTEMS

1. *Development of Linear System Analysis and Idealization of Actual Systems as Linear*

The classical method of solving linear differential equations is well known and furnishes a great urge to the idealization of actual physical systems as linear. Thus in a mechanical system with a mass and a spring, a linear differential equation of the second order can be formulated, with M denoting mass, B denoting friction, and K denoting "spring constant." We have

$$Mx'' + Bx' + Kx = f \tag{5}$$

where x denotes displacement, $x' = dx/dt$ denotes velocity, $x'' = d^2x/dt^2$ denotes acceleration, and f denotes the forcing function or the applied force. Let the forcing function be a unit step, $x(0) = 0$, and $x'(0) = 0$. The solution is

$$x(t) = \frac{1}{K} + \frac{1}{M(s_1 - s_2)}\left[\frac{\epsilon^{s_1 t}}{s_1} - \frac{\epsilon^{s_2 t}}{s_2}\right] \tag{6}$$

where s_1 and s_2 are the roots of the characteristic equations $Ms^2 + Bs + K = 0$.

Heaviside's operational calculus was developed by Carson (8), Bush (9), Berg (10), Cohen (11), Wagner (11a), and others (11b) in the years 1926–1942. From 1942 on, Laplace transforms (12) have been largely used in the study of feedback control systems. Thus, denoting s as a complex variable, $X(s)$ as the transform of $x(t)$, and

$F(s)$ as the transform of $f(t)$, Eq. 5 is transformed to

$$(Ms^2 + Bs + K)X(s) = F(s) \tag{7}$$

giving

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K} = G(s) \tag{8}$$

where $G(s)$ denotes the *transfer function* of the system.

Many techniques have been developed in the study of feedback control systems. To mention a few: Bode's diagram (13), Nyquist criterion (14), Evans' root-locus method (15), Mason's signal flow graph (16). From 1947 on, control system *synthesis* rigorously calls for a logical procedure for the transition from specifications to system. The Guillemin-Truxal approach (17, 18) proposed the following steps:

1. The closed-loop transfer function is determined from the specifications.
2. The corresponding open-loop transfer function is found.
3. The appropriate compensation networks are synthesized.

It may be mentioned that Fourier transforms (19) and Fourier integrals (20) have been developed for the analysis and synthesis of networks. The frequency-analysis techniques are highly useful in the study of communication systems, such as modulation and random noise.

2. Recognition of Nonlinear Systems and Analysis of Such Systems as They Actually Are

Recognition of nonlinear systems as they are indeed has had a long and glorious history. Astronomers, physicists, and engineering scientists have attacked actual nonlinear problems in celestial mechanics, fluid dynamics, elasticity, electronic circuits, and many other fields with fruitful results. We shall roughly divide the methods of attack in four categories:

1. Perturbation and iteration methods.
2. Isocline method and other graphical methods.
3. Linearization methods.
4. Phase-space method, and the use of analog and digital computers.

The perturbation method was developed by H. Poincaré (21) and A. Lindsteht (22) primarily for astronomical problems. Poincaré's *Les méthodes nouvelles de la mécanique céleste* and Liapounov's *Problème général de la stabilité du mouvement* (23) are two classics which appeared in 1892–3. Lord Rayleigh's *Theory of Sound* (24) was published in 1893–96, in which nonlinear problems were discussed.

The term "isoclinal lines" appeared in a paper by J. H. M. Wedderburn (25) published in the Royal Society Proceedings at Edinburgh in 1902. The isocline method was used by van der Pol (26) to solve Eq. 4 in 1921–6. In 1937 Andronow and Chaikin (27) published their *Theory of Oscillations* in which the phase plane and the method of isoclines were described. They referred to K. Runge's *Craphische Methoden* (28) published in 1932 and remarked: "The best graphical procedure is the *method of isoclines*," although they were aware of Liénard's construction (29) made known in 1928. The work of the Soviet Institute of Oscillations founded by L. I. Madelstam, of which Andronow and Chaikin were members, has become known to the world at large through N. Minorsky (30). Other graphical methods are developed by Ku (31), Hsia (32), Paynter (33), and Buland (34).

The third category includes the work of Kryloff and Bogoliuboff (35) and the recent work of Goldfarb (36), Tustin (37), Oppelt (38), Kochenburger (39), Johnson (40) and others. Based upon Professor Lefschetz's English translation, we know that Kryloff and Bogoliuboff considered primarily equations of the form

$$x'' + \omega^2 x = \epsilon f(t, x, x', \epsilon) \tag{9}$$

where ϵ is a small positive quantity and f is a power series in ϵ, whose coefficients are polynomials in x, x', $\sin t$, $\cos t$. (As a matter of fact, generally f contains neither ϵ nor t.) Quoting Lefschetz's Introduction: "Similar equations are well known in astronomy and have been the object of systematic investigation by Lindstedt, Gyldén, Liapounoff and, above all, by Poincaré. In a general sense, one may say that the same methods are applied by Kryloff and Bogoliuboff. . . . The solutions are approximated by the first n terms of certain asymptotic representations; the first two terms usually suffice and yield what the authors call the 'refined first approximation' which they discuss at length."

In Chapters V and VI of reference 35, the method of *linearization*[3] is described, which frequently enables one to by-pass the differential equation and proceed directly from the physical problem to the approximate solutions. Rewriting Eq. 9 as

$$mx'' + kx + \epsilon f(x, x') = 0 \tag{10}$$

where m and k are positive constants, and obtaining as the first approximation a solution $x = a \cos \psi$, where a and ψ satisfy certain conditions, we may say that the first approximation satisfies to within the order ϵ^2 the linear equation

$$mx'' + \bar{\lambda}x' + \bar{k}x = 0 \tag{11}$$

[3] See also Chapter XII of ref. 30 on "method of equivalent linearization" and "principle of harmonic balance."

where, for $\nu = \sqrt{k/m}$,

$$\bar{\lambda} = \frac{-\epsilon}{a\nu\pi} \int_0^{2\pi} f(a \cos \phi, -a\nu \sin \phi) \sin \phi \, d\phi \tag{12}$$

$$\bar{k} = k + \frac{\epsilon}{a\pi} \int_0^{2\pi} f(a \cos \phi, -a\nu \sin \phi) \cos \phi \, d\phi. \tag{13}$$

Thus the nonlinear restoring force term in Eq. 10 is replaced by $\bar{\lambda}x' + k_1 x$, where $k_1 = \bar{k} - k$. It can be remarked also that $\delta = \bar{\lambda}/2m$ is the dissipation decrement in the equivalent linear circuit and $\omega = \sqrt{\bar{k}/m}$ the proper period of its oscillations, to within the order of ϵ^2.

Kochenburger's Describing Function Method (39) of analyzing nonlinear systems is based on three assumptions:

1. There is only one nonlinear element in the system.
2. The output of the nonlinear element depends only on the present value and past history of the input. No time-varying characteristics are included in the nonlinear element.
3. If the input of the nonlinear element is a sinusoidal signal, only the fundamental component of the output is considered.

The approximation involved in the describing-function method is similar to that used by Kryloff and Bogoliuboff. Independent developments of essentially the same methods were made by L. C. Golfarb (36) in U.S.S.R., by Tustin (37) in Great Britain, and by W. Oppelt (38) in Germany.

According to Tustin (37):

> It is evident that if the system does in fact maintain a self-sustained oscillation, the periodic variation of A (the input) must have a waveform such that, when distorted by the nonlinear relationship $B = N(A)$ and when further operated upon by the remainder of the loop transmission $Y(p)$, the waveform A is again produced, with the same fundamental and the same harmonics.

Though this procedure of neglecting the effect of the harmonics produced by a nonlinearity cannot "be justified on any general grounds," it may be justified in particular cases in consideration of the actual type of motion involved when self-sustained oscillation occurs.

We shall review the phase-space method briefly. According to Andronow and Chaikin (27):

> Generally speaking a dynamical system with n degrees of freedom depends upon a certain number n of positional coordinates $q_1, \cdots, q_n$ and the state of the system at time t is fixed by the values of the q_i and their velocities q_i'. We may consider the q_i, q_i' as coordinates of a space S of $2n$ dimensions called the *phase space*. To each state of the system there corresponds the point M with coordinates q_i, q_i' in the space S and as t varies the point M describes a curve called a *path*, which describes the history of the system. . . . The totality of all the paths may be described as the *phase portrait* of the system. It represents all the possible histories of the system; any one of them is determined by a single state M_0. . . ."

Although the phase-plane method is well known in the solution of second-order nonlinear differential equations, the actual solution of high-order nonlinear differential equations depends on the extension from the phase-plane method to the phase-space method (41–45). It may be pointed out that the phase-space "portrait" is a representation of a dynamic system instead of a graphical method. Either digital or analog computers may be used in connection with the phase-space method.

3. *Introduction of the New Concept of Nonlinear Control and Improvement of the Performance of Physical Systems*

The new concept of nonlinear control can be stated in general terms as follows (45):

> A feedback system can be controlled either linearly or nonlinearly. If the system has inherent nonlinear characteristics, it is possible to introduce linear compensating networks to improve the system performance. If optimum system performance desired in a control system cannot be obtained by a combination of linear transfer functions, it is most desirable to introduce nonlinearities into the control system. If the system has inherent nonlinear characteristics, linear compensation and nonlinear control can both be introduced. In short, the newest aspect of feedback control is development of the theory of nonlinear control. . . .

While a physical system with a given input has to be analyzed to know its response or synthesized to yield the desired output, a general nonlinear system with random inputs offers even greater challenge for analysis and synthesis. We shall here describe the Wiener-Lee-Bose approach (7, 46–48) briefly as follows.

According to Wiener (7), the most general probe for the investigation of nonlinear systems is gaussian noise with a flat power-density spectrum, because there is a finite probability that this noise will, at some time, approximate any given time function arbitrarily closely over any finite time interval. Now one can *characterize* a nonlinear system by its response to shot-noise excitation just as one characterizes a linear system by its response to a unit impulse or a unit step or a trigonometric input. The shot effect is to be characterized by a particular set of orthogonal functions, namely, the Laguerre functions (46). In Fig. 3, the outputs of the Laguerre networks are denoted by the Laguerre coefficients $u_1, u_2, \cdots, u_n, \cdots$. The outputs have the following interesting properties: (1) they are gaussianly distributed; (2) they are statistically independent; and (3) they all have the same variance. Since the output $y(t)$ of a nonlinear system depends on the past of the input $x(t)$, which is characterized by the Laguerre coefficients, $y(t)$ can be expressed as

$$y(t) = F[u_1, u_2, \cdots, u_n, \cdots]. \tag{14}$$

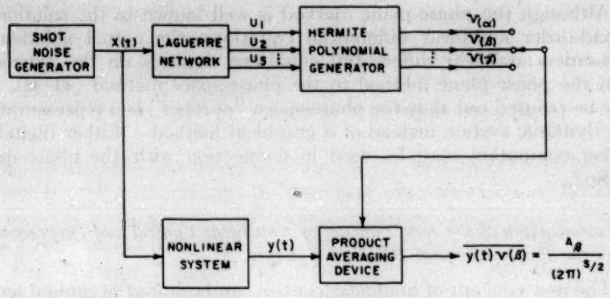

FIG. 3. Block diagram of the circuitry for the characterization of nonlinear systems.

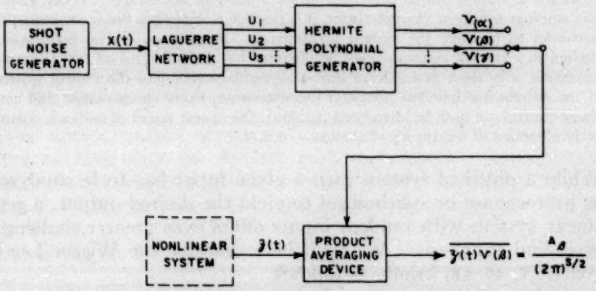

FIG. 4. Block diagram of the circuitry for the determination of the optimum system coefficients.

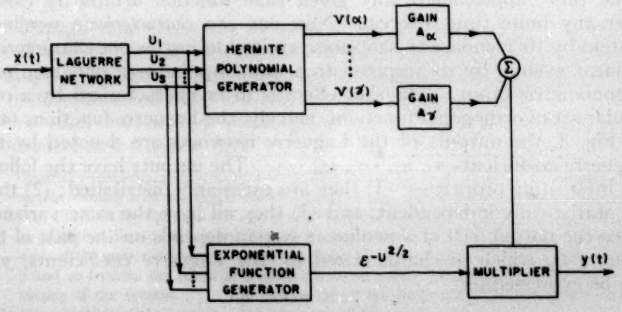

FIG. 5. Block diagram of the circuitry for the synthesis of nonlinear systems.

The Hermite functions are chosen as the expansion for the function F because they form a complete orthogonal set over the interval $-\infty$ to ∞, and are particularly adapted to a gaussian distribution. Equation 14 can then be given in the form

$$y(t) = \lim_{s \to \infty} \sum_{\alpha} A_\alpha V(\alpha) \epsilon^{-u^2/2} \tag{15}$$

where $u^2 = u_1^2 + u_2^2 + \cdots + u_s^2 + \cdots$, $V(\alpha)$ represents the output of Hermite polynomial generator, and A_α represents the *characteristic coefficient* of the nonlinear system, as shown in Fig. 3. In the practical case, we use a finite number of Laguerre coefficients and Hermite functions. Thus the limit $s \to \infty$ may be left out. Using the least weighted mean square error criterion, we get[4]

$$(2\pi)^{s/2} \overline{y(t) V(\alpha)} = A_\alpha \tag{16}$$

which provides the basis for the experimental determination of the characteristic coefficients A_α. In getting Eq. 16, we have replaced the time average by the corresponding ensemble average according to the ergodic theorem.

The experimental procedure is shown in Fig. 3, as given by Bose (**47**, **48**). Since $y(t)$, in an ideal case, should approach the desired output, $z(t)$, the experiment can be carried out by taking the averages of $z(t) V(\alpha)$ instead of $y(t) V(\alpha)$. Thus we can have the procedure as shown in Fig. 4. After the characteristic coefficients A_α are determined from the experiment shown in Fig. 4, a synthesis procedure can be developed according to Eq. 15 with s finite, as shown in Fig. 5.

The Wiener-Lee-Bose procedure has been developed for the analysis and synthesis of optimum nonlinear filters. However, the method is applicable to the study of nonlinear control systems with random inputs. It is therefore possible to analyze or synthesize nonlinear control systems for random inputs and desired outputs, once one has decided to use nonlinear control for the improvement of the performance of physical systems.

III. LINEARIZATION TECHNIQUES

1. Development of the Describing Function Method

We have already mentioned the contributions of Goldfarb, Tustin, Oppelt, and Kochenburger in the development of the describing function method. E. C. Johnson (**40**) applied the sinusoidal analysis to study the free play or backlash in servomechanisms. While in the contactor servomechanism Kochenburger chooses a linear compensating network, in the backlash case Johnson selects a suitable nonlinear mechanical linkage to fit into the system.

[4] $\overline{y(t) V(\alpha)}$ denotes the time average of $y(T) V(\alpha)$.

Use of describing function representation of nonlinear elements substitutes a *quasi-linear* system whose parameters change slowly for one containing rapidly varying relationships. The new system with slowly changing parameters can be thought of in terms of the poles and zeros of its response functions. As amplitude and frequency shift, the poles and zeros of the hypothetical system move about in the complex plane. Thus stability of the system can be studied similarly as in the linear case. One can tell whether the hypothetical system is stable or unstable, or has periodic oscillation. However, as both Johnson and Nichols (49) pointed out, the low-frequency intersecting point of two loci is a divergent equilibrium point, while the high-frequency intersecting point is a convergent equilibrium point. The low-frequency divergent limit cycle prediction must be disregarded if the describing function method is to give predictions that check with exact analysis and actual physical experiments.

At the Heidelberg Conference 1956, J. C. West (50) discussed the use of frequency response analysis in nonlinear control systems. K. Klotter (51, 52) at the same time offered a paper on Hamilton Describing Function in contrast to Kochenburger's Fourier Describing Function. There have been so many papers using the describing function technique in recent years that we can say this method has been generally accepted as a useful tool in the study of nonlinear systems.

2. Other Linearizing Techniques

We shall briefly mention three examples: (*a*) quasi-linearization; (*b*) equivalent linearization; and (*c*) piecewise linearization.

The quasi-linearization technique developed by K. Chen (53) can be used for transient study of nonlinear feedback control systems. Booton's quasi-linearization method (54) for analysis of nonlinear systems with random inputs will be discussed later. Based upon the principle of quasi-linearization (53) that "the quasi-linear representation of a nonlinear element should be obtained by testing the element with its actual input," a nonlinear element for transient study should be obtained by testing the nonlinearity with a transient type of signal which simulates the actual input. Assuming that the nonlinearity is piecewise linear, the operating point of the nonlinear element is expected to shift during the transient over several linear ranges. A quick evaluation of the actual transient response in the first linear range through linear servo techniques determines the test function for the nonlinearity in the first two linear ranges. The quasi-linearization representation thus obtained permits a quick evaluation of the actual transient response in the second linear range. This will in turn determine the test function for the nonlinearity in the first three linear ranges, and so forth.

R. W. Bass (55) proposed the equivalent linearization which can be explained in a houristic way as follows. Consider the difference-differential equation

$$Ax'''(t + t_d) + x''(t + t_d) + Bx'(t + t_d) + Cx(t + t_d) + F[x(t), x'(t)] = 0 \quad (17)$$

where F denotes a nonlinear function of $x(t)$ and its derivative $x'(t)$, A, B, C are constants, and t_d denotes a time-lead. Note that the original equation might have a nonlinear function of $x(t - t_d)$ and $x'(t - t_d)$, where t_d denotes a time-lag. The steps are: (1) insert the complex solution $x(t) = ae^{jvt}$ in Eq. 17 to obtain the formal "instantaneous characteristic polynomial"; (2) replace complex numbers in the arguments of all nonlinear terms by their real parts, and weight each of these nonlinear terms by a factor of ?; (3) divide the polynomial by ae^{jvt}; (4) multiply it by $d(vt)$ and average over a period (from 0 to 2π); (5) separate this averaged polynomial into real and imaginary parts and find the roots a, v of the resultant pair of real equations. Then one can study whether each solution (a_k, v_k) is stable or unstable as in the studies of E. C. Johnson (40, 56).

Piecewise linearization has a long history. However, T. E. Stern (57) has developed piecewise-linear network analysis and synthesis with the view of bridging the gap between the very general studies of nonlinear systems, exemplified by the work of Wiener (7), Zadeh (58), and others. A systematic formulation of the problem leads to a suitable symbolism and an algebra of inequalities (59). This formulation helps to broaden the scope of diode network synthesis in particular and of piecewise-linear synthesis in general. It is of interest to note that Mason and Zimmermann (60) have used this piecewise-linear method in the teaching of electronic circuits.

IV. THE PHASE-SPACE METHOD AND RELATED CONCEPTS

1. *The Phase-Space Method as a General Formulation of the Nonlinear Physical System*

As mentioned before, a dynamical system with n degrees of freedom depends on n positional coordinates q_i, and the state of the system at any time depends upon the values q_i and their velocities q_i'. Thus a ***phase-space*** of $2n$ dimensions is associated with the behavior of the dynamical system. By choosing a system of coordinates we do not necessarily have to solve the problem graphically. However, the usual association of the "isocline method" with the solution of a phase-plane equation suggests a close relationship between the phase portrait and the analytic solution.

In discussing K. Chen's paper (53), R. E. Kalman said:

> It should be strongly emphasized here that the phase-plane, or rather, phase space is not just a name for a clever method directed toward solving particular problems, but it is the most fundamental concept of the entire theory of ordinary differential equations (linear or nonlinear, time-dependent or time-invariant) at the present time. This is because the phase-space is a way of representing (sometimes only in an abstract sense) all possible solutions of a differential equation. . . .

2. *Method of Simultaneous Phase-Plane Equations*

In "Analysis of Nonlinear Coupled Circuits—II" (43), the author developed the method of simultaneous phase-plane equations (see Section 10–6, ref. 45, pp. 234–243). For a system of *three* degrees of freedom, the phase space is characterized by *six* coordinates—three positional coordinates q_i and three velocity coordinates q_i'. In an electric network of three loops, the phase-space coordinates represent three charges and three currents. In a coupled circuit of two loops, the phase-space coordinates are reduced to two charges and two currents. Though this method is developed later than the method of high-order phase-space equation, it is logical from the viewpoint of the phase-space representation of the state of a general dynamical system.

As shown in references 61 and 62, the method of simultaneous phase-plane equations is highly powerful in the analysis of feedback control systems with nonlinearity either in the forward branch or in the feedback branch or in both. Taking a two-loop system for illustration, the phase-space equations are

$$\frac{dq_1'}{dq_1} = \frac{q_1''}{q_1'} \tag{18}$$

$$\frac{dq_2'}{dq_2} = \frac{q_2''}{q_2'} \tag{19}$$

where q_1'' and q_2'' denote respectively the second derivatives of q_1 and q_2 with respect to time, and they are in general functions of q_i and q_i'.

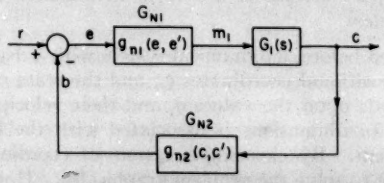

FIG. 6. Block diagram of a nonlinear control system.

In a nonlinear control system shown in Fig. 6 (see Section 12–9, ref. 45, pp. 303–305), there are two nonlinear controllers G_{N1} and G_{N2} in the

forward branch and the feedback branch respectively. The two variables are e and c, corresponding to the positional coordinates q_1 and q_2. The other two coordinates in the phase-space are e' and c', corresponding to q_1' and q_2'. The differential equations are

$$e'' + g_{n1}(e', e) = m_1(t) = Tc''' + c'' \tag{20}$$

$$c'' + g_{n2}(c', c) = b(t) = r - e \tag{21}$$

where r denotes input or reference signal, c denotes output, e denotes actuating signal, b denotes the feedback signal, m_1 denotes the control effort, T denotes the time constant, and g_{n1} and g_{n2} represent two nonlinear functions. The c''' term on the righthand side of Eq. 20 can be eliminated by differentiating Eq. 21 with respect to time and substituting in Eq. 20. Thus we have

$$e'' + g_{n1}(e', e) + Te' + e = f(r', r, c'', c', c). \tag{22}$$

The phase-plane equations corresponding to Eqs. 18 and 19 are

$$\frac{de'}{de} = \frac{e''}{e'} = \frac{f(r', r, c'', c', c) - g_{n1}(e', e) - Te' - e}{e'} \tag{23}$$

$$\frac{dc'}{dc} = \frac{c''}{c'} = \frac{r - e - g_{n2}(c', c)}{c'}. \tag{24}$$

Note that c'' in Eq. 23 can be expressed in terms of $(r - e - g_{n2})$ just as in Eq. 24.

The simultaneous solution of these equations calls for numerical techniques or the help of computers. The general formulation is exact and complete. The nonlinear functions can be given either in analytic form or in graphical form. It is hoped that the programming of phase-space equations will help the study of nonlinear control systems.

3. *Phase-Space Approach to the Design of Saturating Servomechanisms*

We shall cite two concrete examples about the phase-space approach to the design of servomechanisms. R. E. Kalman (**63, 64**) made a phase-plane analysis of automatic control systems with nonlinear gain elements and then developed a method for the design of second and higher order saturating servomechanisms. Kalman's method is based on linear transformations in the phase space, corresponding with the partial-fraction expansion of transfer functions to separate natural frequencies, and makes use of the root-locus method for the qualitative study of close-loop stability. The analysis leads to replacing a high-order system with a second-order system which closely approximates

the former. The method is applied to the study and design of saturating servomechanisms. The principle of analysis is based on the following:

> The trajectories of a nonlinear differential equation may be approximated with arbitrary accuracy by breaking up the phase space into a sufficiently large number of regions, such that within each region the trajectories obey a linear differential equation with constant coefficients. This is equivalent to approximating nonlinear characteristic curves by straight-line segments. . . .

An earlier paper by A. M. Hopkin (65) used the phase-plane approach in the compensation of saturating servomechanisms. A new device is developed and called an *anticipator*. It controls the application and reversal of the manipulated variable. It senses the point where the actual system phase-plane trajectory crosses the particular trajectory making up the critical boundary, that is, one of the two which go into the phase-plane origin. It there reverses the manipulated variable to bring the system operating point along the critical boundary to zero velocity with zero error. In this study the optimum performance is assumed to be that behavior in which the system returns to rest with zero or minimum error in the shortest time following a step input. The anticipator unit utilizes a biased-diode nonlinear element. For a constant-velocity input, a narrow-proportional-band amplifier is suggested as the nonlinear network.

A recent paper by Hopkin and Iwama (66) reported about the design of a predictor-type air-frame controller by phase-space analysis. As the method of calculating the phase-space trajectories from analytical expressions for e, e', and b (the feedback signal to the predictor controller) is exceedingly tedious, most of the trajectories are obtained by the technique of "reversing the time axis" with the use of an analog computer. For this study made on an analog computer, a workable solution is found in the form of a cathode-ray oscillograph-photocell combination with a mask which changes shape as b varies to simulate the effect of the varying shape of the projections of the critical boundary surfaces with b. The particular predictor is set up to handle the case where the controlled system is of third order with one highly oscillatory mode. Thus a control system of the optimum relay type is obtained to control the linear pitch acceleration of an air frame and to compensate automatically for parameter changes resulting from changing flight conditions.

4. *Switching Criteria for High Order Servomechanisms*

In the field of contactor or on-off servos, phase-plane techniques are used by MacColl (67) and Weiss (68) to determine the response and improve the stability of second-order servos possessing such nonlinearities as finite torque reversal time and coulomb damping. D. C.

McDonald (69), A. M. Hopkin (65), N. S. Neiswander and R. H. MacNeal (70), however, used the phase-plane technique to optimize performance. In McDonald's method a torque-reversing criterion is developed in which the sign of the torque applied to the load is determined by a nonlinear function of error and error rate. I. Bogner and L. F. Kazda (71) extended McDonald's method to obtain unique switching criteria for contactor servomechanisms of order greater than two.

Bogner and Kazda's analysis presupposes a discontinuous forcing function, fixed in magnitude and reversible in sign, supplied to a dynamic system the operation of which is describable over every interval by a linear differential equation. This type of control, while nonlinear in the general sense of the term, is sectionally or piecewise linear. The method makes use of the particular phase space in which servo error is plotted against its higher derivatives. It is of interest to note that this phase space having the variable and its higher derivatives as the coordinates[5] corresponds to the phase space used in the author's method of high-order phase space equation (see Section 10–5 of ref. 45, and refs. 42–44). Let the third-order differential equation be given by

$$e''' + e'' = \delta \tag{25}$$

where δ denotes either position or velocity input, positive or negative in sign. The *three* phase-space coordinates are: $u_1 = e$, $u_2 = e'$, and $u_3 = e''$. In discussing the switching criteria, Bogner and Kazda made use of the *principal coordinate* phase-space, where the new coordinates are: $v_1 = u_1 - u_3$, $v_2 = u_1 + u_3$, and $v_3 = u_2$. In an isometric drawing of the v-space trajectory for a third-order contractor servo, a representative point moves from 0 to 1, where it intersects the first switching surface. The sign of the controlled variable is then reversed, permitting the representative point to move in the surface until it reaches the second switching point 2. It then follows the trajectory into the origin.

N. J. Rose (72) used a more complicated switching rule, with the correcting force depending on both the error and the error rate. Thus it is possible to find the optimum switching rule which reduces the error and the error rate simultaneously to zero in a minimum time. The optimum switching criteria given by Rose apply to second and higher order systems.

S. S. L. Chang (73) discussed the optimum switching criteria for higher order contactor servo with interrupted circuits and asked the question whether the assumption that the error and its derivatives up

[5] This phase space is a consequence of Cauchy's Theorem of Existence for the solution of a differential equation, as a differential equation of the n^{th} order is equivalent to a system of n equations of the first order.

to the $(n-1)^{th}$ order should be continuous at the moment of switching is necessary or not. He investigated the optimum open-circuit switching criteria for a high-order contractor servo by an extension of the phase-space method. Hung and Chang (74) extended the optimum switching theory to a more general form such that phase space analysis is applicable to a system which has non-vanishing power of s in the numerator of its transfer function. They concluded: "The phase-space method is a basic way of thinking in analyzing nonlinear system."

V. NEW ANALYTIC TECHNIQUES AND NEW TRANSFORMS

1. Reversion Method and Iteration Method

L. A. Pipes has applied operational methods (75) to the analysis of nonlinear systems. The reversion method (76) is based on the algebraic procedure used in reverting power series. Let the nonlinear differential equation be written in the form

$$a_1 y + a_2 y^2 + \cdots + a_5 y^5 + \cdots = k\,\phi(t) \tag{26}$$

where the a_i coefficients are, in general, operators and functions of the operator $D = d/dt$. Assume a solution of the form

$$y = A_1 k + A_2 k^2 + A_3 k^3 + \cdots. \tag{27}$$

The A_i coefficients may be determined by substituting Eq. 27 into Eq. 26 and equating coefficients of equal powers of k. This method can be compared with the method of Picard. Whereas the reversion method is used to analyze a series circuit with a nonlinear inductor, Pipes introduced the minimum mean square error method in the analysis of a series circuit with a nonlinear capacitor (77). It may be noted that the author used the acceleration-plane method to analyze a series circuit with both nonlinear inductance and nonlinear capacitance (78).

Lalesco's nonlinear integral equation was also explored by Pipes (79). In the study of forced oscillations of nonlinear circuits, the method of undetermined coefficients was extended (80). For circuits with time-varying parameters, the Brillouin-Wentzel-Kramers method was suggested (81). An iterative method (82) for nonlinear vibrations was given by R. E. Roberson who earlier contributed a paper on the synthesis of a nonlinear dynamic vibration absorber (82a).

2. Number Series Transformation Method

The number series transformation method was introduced by Tustin in 1947 (83). A. Madwed (84) made further developments and gave the following advantages of this method: (1) the problems involved in analyzing and synthesizing systems are solved in the time domain;

(2) the method permits the analysis and synthesis of nonlinear systems; (3) the actual mathematical operations involved are simple addition, subtraction, multiplication and division of real numbers; and (4) the method is easy because it is an *operational* method. It is shown (**84a**) that it is possible to take the Fourier and Laplace transforms of the number series transformations of functions.

3. Harmonic Analysis and Periodic Solutions

In order to justify equivalent linearization, Kryloff and Bogoliuboff (**35, 30**) observe that a nonlinear oscillatory process is generally characterized by a certain Fourier spectrum of the component frequencies resulting from the nonlinearity of the system. The principle of *harmonic balance* is assumed *a priori*, so that the equivalent parameters can be determined in the linearized system which gives the same oscillation as the fundamental harmonic of the quasi-linear system. L. J. Lewis (**85**) suggested the use of polynomial functions for carrying out the harmonic analysis instead of the more conventional use of trigonometric functions. In discussing this paper, Stout remarked: "In addition to the suggested applications to nonlinear differential equations, modulation, and distortion, it should be useful for the calculation of 'describing functions' employed in nonlinear feed-back system analysis." It may be noted that Lewis' polynomial functions are closely related to the Tschebyscheff polynomials.

Given a modified form of the van der Pol equation

$$x'' - (\epsilon + \mu kx - \mu^2 x^2)x' + x = 2B\omega_1 \sin \omega_1 t \tag{28}$$

where ϵ and μ are parameters which are assumed to be small, and μ is of the order ϵ. A. W. Gillies (**86**) discussed the periodic solutions of the above equation which differs from Eq. 4 by the presence of a forcing function and a relatively large term μkx in the damping coefficient. The effect of this is that the solution has to be developed in powers of μ, and the equations of the first approximation cannot be obtained without considering the second harmonic. Consequently methods such as those of Kryloff and Bogoliuboff or Cartwight's difference equation (**87**) have to be carried a stage farther than is necessary for the van der Pol equation. In an earlier paper (**88**), Gillies used the power series in the study of nonlinear circuits in which a triode is associated with a linear network of any form. Part 1 deals with a stable circuit having zero or negative feedback; Part 2 deals with the free oscillations of an unstable regenerative circuit; and Part 3 deals with the forced oscillations of a regenerative triode circuit. It is shown that the mathematical techniques applied in Parts 1 and 2 can be applied in Part 3, and the uniform procedure leads to a physical description of the phenomena in terms of modulation products.

4. Step-by-Step Method

Many experts on numerical analysis have worked on the step-by-step method. The high speed electronic digital computer uses the step-by-step method. Whether one gets solutions of differential equations in the time domain or one solves phase-space equations, a step-by-step procedure can always be resorted to. This procedure is also used in many graphical methods.

We shall mention as an example a step-by-step method developed by T. M. Stout (**89**) for transient analysis of feedback systems with one nonlinear element.[6] The method is somewhat similar to methods described by Tustin (**83**, **90**), Madwed (**84**), Wagner (**91**), and Truxal (**92**). Unlike these methods, Stout's method does not require resolution of the variables into step, pulse, or other component signals, or the substitution of difference for differential operators; no attempt is made to use regression formulas or z-transform techniques. The weighting function required in the analysis is determined by inverse Laplace transformation of the appropriate transfer function, and the superposition integral is evaluated by standard numerical methods.

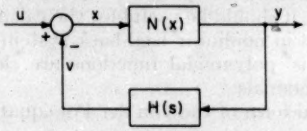

FIG. 7. Block diagram of a system with one nonlinear element.

Referring to Fig. 7, where $y(t)$ is a nonlinear function of $x(t)$ and can be expressed as $y = N(x)$, we have

$$x(t) = u(t) - v(t) = u(t) - \int_0^t h(t-\tau)N[x(\tau)]d\tau. \qquad (29)$$

Note that the weighting function $h(t)$ is the impulse response of the feedback block and is the inverse Laplace transform of $H(s)$. An approximate solution is found as follows: Using numerical methods, the various system variables are computed at intervals $\Delta t = T$; these values are denoted by $v_n = v(nT)$, $h_n = h(nT)$, and so forth. If the trapezoidal rule is used, the superposition integral is approximated by

$$v_n = T[\tfrac{1}{2}h_n y_0 + h_{n-1}y_1 + h_{n-2}y_2 + \cdots + h_2 y_{n-2} + h_1 y_{n-1} + \tfrac{1}{2}h_0 y_n]. \qquad (30)$$

If we let $\beta_n = Th_n$, Eq. 30 can be regarded as the product of the two sequences $y_0/2$, y_1, $\cdots$, y_n and $\beta_0/2$, β_1, $\cdots$, β_n. The first value of

[6] See also Naumov's method (ref. **89b**) and Dietz's method (ref. **89c**).

$y(t)$, y_0, is found from the initial conditions; the rest of the values are determined step-by-step.

5. Complex Convolution Method

The method is developed by E. Weber (93). The Laplace transform of the product $g(t)h(t)$ is given by the complex convolution integral

$$\mathfrak{L}[g(t)h(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} G(s-q)H(q)dq = G(s)^*H(s) \tag{31}$$

where the last term is a symbolic notation. Similarly, we have

$$\mathfrak{L}[h(t)h(t)] = H(s)^*H(s) = H^{(2)}(s) \tag{32}$$

and

$$\mathfrak{L}[h^n(t)] = H^{(n)}(s). \tag{33}$$

Taking for example the differential equation

$$y'' + a_1y' + a_0(y + ky^n) = f(t) \tag{34}$$

and applying the Laplace transform, we obtain

$$(s^2+a_1s+a_0)Y(s)+a_0kY^{(n)}(s) = F(s)+(s+a_1)y_0+y_0' = F_1(s) \tag{35}$$

where $Y(s)$ and $F(s)$ are the Laplace transforms of $y(t)$ and $f(t)$, respectively, s is the complex variable, a_1 and a_0 are constants, y_0 and y_0' are the initial values of y and y', respectively. Assume "rather boldly" that $Y(s)$ be a mermorphic function and write it

$$Y(s) = \sum_\alpha \frac{A_\alpha}{s - s_\alpha} \qquad \alpha = 0, 1, 2, \cdots. \tag{36}$$

For $n = 3$, we get

$$Y^{(3)}(s) = \sum_\alpha \sum_\beta \sum_\gamma \frac{A_\alpha A_\beta A_\gamma}{s - (s_\alpha + s_\beta + s_\gamma)}. \tag{37}$$

It has been assumed that there exists the Laplace transform of $y(t)$. If $Y(s)$ is mermorphic, then the Laplace transform of any integer power of the function is again mermorphic. Substituting Eqs. 36 and 37 in Eq. 35 gives, for $F(s) = 1/s$ and $n = 3$,

$$\sum_\alpha \frac{A_\alpha}{s - s_\alpha} + \frac{a_0k}{(s - s')(s - s'')} \sum_\alpha \sum_\beta \sum_\gamma \frac{A_\alpha A_\beta A_\gamma}{s - s(s_\alpha + s_\beta + s_\gamma)} = \frac{1 + s^2y_0 + s(a_1y_0 + y_0')}{s(s - s')(s - s'')} \tag{38}$$

where s' and s'' are the roots of the characteristic equation $s^2 + a_1 s + a_0 = 0$. With further simplification of the triple summation term, Eq. 30 can be reduced to a form where coefficient comparison is possible, provided the unknown s_a's are not identical to s' or s''.

6. Taylor-Cauchy Transform Method

The Taylor-Cauchy transform method (**94**) was developed by Ku, Wolf, and Dietz. The Taylor-Cauchy transform is defined by[7]

$$\mathcal{T}_c[F(\lambda)] = \frac{1}{2\pi j}\int_c \frac{F(\lambda)}{\lambda^{n+1}}\,d\lambda \qquad n = 0, 1, 2, \cdots \tag{39}$$

where λ is a complex variable which replaces the real variable t, and where $F(\lambda)$ replaces any complicated time function, like $D^2y = y''$ and $N(y) = y^3$, by substituting W for y and λ for t, like $D^2W(\lambda) = W''$ and $N(W) = W^3$. Taking Eq. 34 as an illustration, the transform equation is

$$\mathcal{T}_c[(D^2 + a_1 D + a_0)W(\lambda)] + \mathcal{T}_c[a_0 k W^3(\lambda)] = \mathcal{T}_c[F(\lambda)] \tag{40}$$

where $D = d/d\lambda$ and $D^2 = d^2/d\lambda^2$. The new transform is linear in the sense that it has additivity and communitivity properties. In general, the highest derivative term $W^{(k)}(\lambda)$ is assumed to be analytic for a class of systems. Then

$$W^{(k)}(\lambda) = \sum_{n=0}^{\infty} w_{n,k}\lambda^n. \tag{41}$$

Applying the Taylor-Cauchy transform to both members of the above equation gives

$$\mathcal{T}_c[W^{(k)}(\lambda)] = w_{n,k}. \tag{42}$$

The inverse Taylor-Cauchy transform is then given by

$$\mathcal{T}_c^{-1}[w_{n,k}] = \sum_{n=0}^{\infty} w_{n,k}\lambda^n = W^{(k)}(\lambda). \tag{43}$$

The transform of nonlinear terms is discussed in refs. **94**, **95**, and **96**. Three tables of Taylor-Cauchy transforms are given at the end of ref. **94**. Given the simple nonlinear differential equation

$$\frac{dy}{dt} + y^2 = u(t) = \text{unit step}. \tag{44}$$

[7] C denotes a closed contour in the λ-plane enclosing the singularities of $F(\lambda)$.

Taking the Taylor-Cauchy transform gives

$$w_n + C_{n-2,0}^{(2)} = \delta_n \tag{45}$$

in which

$$C_{n-2,0}^{(2)} = \sum_{\beta=0}^{n-2} \frac{w_\beta}{\beta+1} \frac{w_{n-\beta-2}}{n-\beta-1}. \tag{46}$$

Solving Eqs. 45 and 46 for w_n recursively gives $w_0 = 1$, $w_1 = 0$, $w_2 = -1$, $w_3 = 0$, $w_4 = 2/3$, $w_5 = 0$, $w_6 = -17/45$, and so forth. Since w_n's are solved for the first derivative $W'(\lambda)$ in this example, we obtain

$$W(\lambda) = \sum_{n=0}^{\infty} \frac{w_n \lambda^{n+1}}{n+1} + W_0. \tag{47}$$

For $y_0 = 0$ and hence $W_0 = 0$, Eq. 47 becomes

$$W(\lambda) = \lambda - \frac{\lambda^3}{3} + \frac{2}{15}\lambda^5 - \frac{17}{315}\lambda^7 + \cdots = \tanh \lambda \tag{48}$$

which gives $y(t) = \tanh t$ as the solution of Eq. 44.

VI. THEORY OF NONLINEAR SYSTEMS

1. *New Concepts and Theorems*

Some general theorems for nonlinear systems possessing resistance or reactance were given by Millar (**97**) and Cherry (**98**) in 1951. Energy relations in nonlinear inductors and capacitors were discussed by Manley and Rowe (**99**) in 1956. In a paper presented at the Cranfield Conference (**100**), Cherry and Millar gave nine new theorems and explained the following new concepts.

The concept of *content* and *co-content*. Associated with any nonlinear resistor characteristic, the content $G = \int v di$ and co-content $J = \int i dv$ are duals. The sum $(G + J)$ always equals the instantaneous power.

The concept of *energy* and *co-energy*. The energy stored in an inductor is $T = \int i d\phi$ and the energy stored in a capacitor is $U = \int v dq$. The co-energy for an inductor is $T' = \int \phi di$ and the co-energy for a capacitor is $U' = \int q dv$.

We shall quote two Theorems as follows:

Theorem 3. In any all-resistor multi-mesh circuit driven by current sources only, the total content G is unaltered if infinitesimal increments be given to any of the currents in the system, provided these increments are consistent with Kirchhoff's current law. Similarly, in a voltage-driven circuit, the total co-content is stationary with respect to consistent voltage variations.

Theorem 8. In any circuit of voltage (current)-driven nonlinear inductor elements, the total energy T (co-energy T') has a stationary value with respect to node-voltage (mesh-current) variations, provided that these are consistent with Kirchhoff's voltage (current) law.

A dual to Theorem 8 gives Theorem 9, which is obtained by writing capacitor for inductor, node for mesh, voltage for current, charge for flux. In the case of any nonlinear system, possessing dissipative and energy-storing elements, the appropriate Lagrangian function L may be shown to be either $L = T' - U$ or $L = U' - T$, a mixture of energies and co-energies. In terms of current variables Lagrange's equations involve the first form of L and the content G. In terms of voltage variables the equations involve the second form of L and the co-content J. When the networks themselves contain nonlinear elements, nothing resembling Thévenin's theorem applies. However, the use of the terminal characteristic of an active nonlinear resistive network often simplifies graphical work. The new concepts and theorems hold not only for electrical networks, but also for mechanical and electromechanical systems.

According to Manley and Rowe (99), the nonlinear reactive elements may be regarded as modulators in which the input signal frequency and a local oscillator frequency combine to produce new frequencies. These are harmonics and sum and difference combinations of the two applied frequencies, one or more of these resulting sidebands being taken as the useful signal output. The results are thus applicable to nonlinear reactor modulators and demodulators and to magnetic and dielectric amplifiers.

Let the characteristic of a nonlinear capacitor be given by $v = f(q)$, where $f(q)$ is single-valued. If the nonlinear capacitor is connected in a circuit containing two generators of frequencies f_1 and f_0, in general all of the frequencies $f_{m,n} = mf_1 + nf_0$ will be present in the circuit, where m and n take on all integral values, positive, negative, and zero. Assuming that no other frequencies are produced by instability, the charge q, and hence the current i can be written as a double Fourier series, using both positive and negative frequencies. The voltage v across the nonlinear capacitor must consist of the same frequency components as the charge q. Thus v may also be written as a double Fourier series.

Denoting the coefficients of the double Fourier series by $Q_{m,n}$, $I_{m,n}$, and $V_{m,n}$, respectively, for q, i, and v, the average power associated with $(mf_1 + nf_0)$ flowing into the nonlinear capacitor is $W_{m,n} = V_{m,n}I_{m,n}^* + V_{m,n}^*I_{m,n}$, where the asterisk denotes the complex conjugate. Then the general energy relations at the different frequencies are obtained.

$$\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\frac{mW_{m,n}}{mf_1 + nf_0} = 0; \qquad \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{nW_{m,n}}{mf_1 + nf_0} = 0. \tag{49}$$

These relations apply equally well to both the nonlinear capacitor and inductor, the two derivations being almost identical.

2. Stability Theory and New Stability Criteria

An excellent survey of recent work on theory of stability of solutions of nonlinear ordinary differential equations was given by W. Kaplan (**101**) in 1956. An extensive bibliography is appended, the publications being grouped under special topics, and over half of the references are to papers in Russian. Since there is no need of reminding the readers who know Russian about the Russian contributions, refs. **102**, **103**, and **104** are given here as English translations are available. I. G. Malkin's *Methods of Lyapunov and Poincaré in the Theory of Nonlinear Oscillations* was published in Moscow, 1949; and his *Stability of Motion* was published in 1952.[8] While Lyapunov (Liapounoff) first published his paper in 1892, a French translation by E. Davaux appeared in 1907, which was reprinted by the Princeton University Press in 1949. Reprints of Lyapunov's four papers in Russian appeared in 1950. A. I. Lur's published his treatise *Some Nonlinear Problems in the Theory of Automatic Regulation* in 1951. An English translation is now available.

According to Kaplan, for the work outside the Soviet Union, the following are important contributors: Bellman (**105**), Levinson (**106**), Wintner (**107**) in the United States, Miss Cartwright (**108**) in England, Haag (**109**) in France, Cesari (**110**) in Italy and the United States, Massera (**111**) in Uruguay. With regard to stability of forced vibrations, the recent contributions of refs. **112**, **113**, and **114** are worthy of note.

Stability criteria in linear and nonlinear systems have been suggested from time to time. To cite the contributions at the Cranfield Conference 1951, L. S. Dzung (**115**) gave a stability criterion which is "more exact and more generalized" than the Nyquist criterion, R. W. Jones (**116**) applied the method of Liapounoff to give stability criteria for certain nonlinear systems, and A. Leonhard (**117**) proposed "relative damping" as criterion for stability and as an aid in finding the roots of a Hurwitz polynomial. (Some recent German contributions by K. Magnus, L. Cremer, H. Cremer, and F. H. Effertz are listed in refs. **117b**, **c**, **d**, **e**, and **f**.)

For the stability criteria of nonlinear control systems, we shall cite two recent examples. K. Klotter and E. Pinney (**118**) gave a comprehensive stability criterion for forced vibrations in nonlinear systems as follows. Given the equation

$$ax'' + cx + c\,f(x', x) = P\cos\omega_1 t. \tag{50}$$

[8] There are four new Russian books called to the author's attention by Dr. B. Naumov. (See refs. **102b–e**).

Under very weak restrictions it is shown that the stability depends on the sign $\partial A/\partial P$, where A denotes the maximum value of $x(t)$ within a period. The motion is stable if this derivative is positive; it is unstable if it is negative. The plot of A, the maximum value of $x(t)$ *versus* ω_1, the frequency of the driving term, or ω_1^2 is known as a response curve. Once these response curves are established, one can proceed to study whether the motion corresponding to any one point on these curves is a stable one or not. Since Eq. 50 has no known exact solution, one often resorts to methods of integration which provide approximate results. Besides the perturbation and iteration methods, there are the averaging methods, of which the Ritz or Galerkin method is an example (**119**).

A new stability criterion for a class of nonlinear systems was more recently given by Ku and Wolf (**120**). According to the method of partition (**94–96**) the highest derivative $x^{(k)}(t)$ of a k^{th} order nonlinear differential equation is expressible as a summation of $w_{n,k}t^n$, as shown in Eq. 41. The response $x(t)$ is then of the form

$$x(t) = \sum_{n=0}^{\infty} h_n t^n. \tag{51}$$

The Laplace transform of $x(t)$ given by Eq. 51 is

$$X(s) = \sum_{n=0}^{\infty} \frac{h_n n!}{s^{n+1}}. \tag{52}$$

The stability criterion is given as follows. The system is stable if and only if the convex singularity hull (CSH) of $X(s)$ is entirely in the left-half plane. The CSH is constructed by joining the points of singularity of the transform $X(s)$ in the s-plane. It may be noted that the singularities of $X(s)$ are not necessarily poles. The degree of stability is also discussed in connection with the conjugate indicator function, which is called the Phragmen-Lindelof function.

3. Optimum Nonlinear Control Systems

According to Minorsky (**121**), some of the principal ways in which nonlinear relations may be used on control systems are: (1) to compensate for nonlinearities; (2) to control the appearance or disappearance of an oscillation; (3) for frequency de-multiplication; (4) for synchronization; (5) for asynchronous excitation and quenching; and (6) in parametric excitation.

A theory of optimum nonlinear control (**122**), developed by R. Oldenburger, is devoted to systems with the differential equation

$$m' + f(m) = \frac{P(D)e^{-\tau_d D}}{Q(D)}\, g(c) - L(t)\ . \tag{53}$$

where m denotes the deviation in the value of the controlled variable, c denotes the deviation in the value of the controlling variable, $P(D)$ and $Q(D)$ are polynomials in the derivative $D = d/dt$ and with nonzero constant terms, $\epsilon^{-\tau_d D}$ denotes the dead-time operator, $g(c)$ denotes a function of c, $f(m)$ denotes a function of m, and $L(t)$ is a given forcing function of time. Optimum transients for control systems can be obtained by nonlinear control functions depending on m. The results of theoretical studies have been verified experimentally with physical devices and automatically controlled systems.

Design and analog-computer analysis of an optimum third-order nonlinear servomechanisms were reported by Doll and Stout (123). Nonlinear integral compensation of a velocity-lag servomechanism with backlash was made by Shen, Miller, and Nichols (124). Quasi-optimization of relay servos by use of discontinuous damping (125) was also reported. In the field of discontinuous automatic control, I. Flügge-Lotz's treatise (126) was published in 1953. She and Wunch (126a) showed that a system with discontinuous damping and spring coefficients would transfer inputs well, if the change of the coefficients from one constant value to another was controlled by the deviation between input and output. Optimization by the phase-plane technique was given in refs. 70, 71, 72, and 73. Optimum response of discontinuous feedback control systems was analyzed by Nesline (127). A simple nonlinearized control system was given by Zaborszky (128). Cosgriff and Emerling suggested optimizing control systems (129) using logic circuits.

Compensation of feedback-control systems subject to saturation has been extensively studied by Newton (130). Nonlinear compensating networks were recommended by Mishkin and Truxal (131). A nonlinear control system for wide range input signals was reported by Tou and Ku (132). Long and Ham (133) suggested conditional feedback as a new approach to linear and nonlinear feedback control. Compensating transient saturation by excess error storage was analyzed by Chang and Archbald (134).

4. Sampled-Data Systems and Systems with Random Inputs

The vast amount of literature on sampled-data systems cannot be adequately reviewed here (135–7). It can be said, however, that nonlinearities in sampled-data control systems are now studied, and the theory of nonlinear control may well influence the design of sampled-data systems. As examples we may mention the recent papers by Kalman (138), Nease (139), Tou (140), and Mullin (141).

The analysis of sampled-data systems owes much to the pioneer work of Ragazzini and Zadeh (142). Analysis of sampled-data servomechanisms has been done on a digital computer by Tostanoski (143) and on an analog computer by Chestnut (144) and others. It may be of interest to note that Boxer and Thaler (145) made use of the z-transform

in solving linear and nonlinear systems. Tsypkin's *Theory of Intermittent Regulation* has been translated into English (146).

As an introduction to the study of nonlinear systems with random inputs, we like to mention the papers by R. C. Booton, Jr. (54, 147), J. F. Barrett and Coales (148), W. E. Thomson (148b), J. C. West and Nikiforuk (149), K. B. Tuttle (150), D. W. C. Shen (151), K. Chuang and Kazda (151a). In ref. 54, Booton considered an amplitude-sensitive nonlinear element with a Gaussian input and derived a quasi-linearization representation of its response. "When the nonlinearity occurs as part of a feedback system, application of the quasi-linearization method is complicated by the fact that this method assumes knowledge of the probability density function of the input to the nonlinearity. Since, in general, nonlinear operations on an input function change the shape of its probability density function, the inputs to the nonlinearity and to the overall system do not have the same form of probability density function." Even if the input to the nonlinear element is assumed Gaussian, its rms value is unknown. Also the equivalent gain of the element is another unknown parameter, and two relations must be found to determine the equivalent gain. One of these relations is furnished by the quasi-linearization method. The second relation comes from analysis of the over-all quasi-linear system with the equivalent gain treated as a parameter. It is to be noted that the quasi-linear representation of the response is interpreted in terms of statistical characteristics of the actual response, which is described by the autocorrelation of the response and by the cross-correlation between the input and the response.

Wiener's theory (7) has been studied by Singleton (152), Ikehara (152b), Weinberg and Kraft (152c). The recent work of Bose (47, 48) already mentioned in Sec. II-3 of this paper, is a further development of Wiener's theory of nonlinear systems. Zadeh's contribution to the theory of nonlinear systems (58, 153) is a parallel development and deals with optimum nonlinear filters. In ref. 58, it is shown that a general nonlinear two-pole of class N_1 is completely characterized by its responses to a family of step functions with amplitudes ranging over all real values. Several possible modes of realization of two-poles of class N_1 are indicated. It is noted that, in order to determine the optimum filter within the Class N_1, it is necessary to know the second order probability density functions for the signal and noise. Optimization of nonlinear filters is given in ref. 153, and the extension of the theory to nonlinear multipoles is given in ref. 153a.

Zadeh remarked rather optimistically (58):

> The theory of nonlinear systems has made some progress recently toward the formulation of general theorems bearing on both the internal and external behavior of broad classes of nonlinear systems.

In making the above remark, he had in mind refs. **97**, **98**, **152a**, **154**, and **154a**. We might add ref. **155** to this group of papers.

In connection with the optimum nonlinear filter, Zadeh (**153b**) pointed out two approaches: (1) one approach parallels Wiener's approach (to linear systems) by choosing the form or class of filters and then finding the optimum number of this class by minimizing the mean square error between the desired output and the actual system output; (2) the other approach formulates an appropriate statistical criterion and then determines the optimum filter for this criterion with little or no restrictions upon the form of the filter. Both of these approaches yield equations for optimum filters in terms of higher-order statistics (distribution functions and correlation functions) of the input and the desired output.

The new approach uses a polynomial functional method of characterization for the analysis and synthesis of nonlinear systems excited by white noise. This is the method of Wiener (**7**), Bose (**48**), and Barrett (**148**). This approach assumes that we have at our disposal an *ensemble* member of the input time function $x(t)$ and the corresponding *ensemble* member of the desired output $z(t)$. When these ensembles are available, the new approach makes measurements on them that yield optimum nonlinear systems (filters) directly, instead of first measuring the distributions and then solving design equations in terms of these measured values.

Thus we see in the study of nonlinear systems with random inputs, we have started with information theory and correlation functions and now reached a method using ensembles and polynomial functions. We shall mention also that an Ensemble-Transform method has been recently developed by Ku and Wolf (**156**) for the study of nonlinear systems with random inputs (**157**). This method can be used with Taylor-Cauchy transforms for nonlinear systems or Laurent-Cauchy transforms for systems that can be described by differential-difference and sum equations.

VII. CONCLUDING REMARKS

From the above sections we see that the theory of nonlinear control systems has been developed rapidly in the recent years. It is the author's hope that by reviewing the recent advances and interchanging new ideas at the first International Congress of IFAC on Automatic Control further progress can be promoted.

Acknowledgment

The author wishes to acknowledge the encouragement of Dr. J. G. Brainerd, Director of the Moore School, University of Pennsylvania, and the support of the National Science Foundation, Washington D. C. in

the preparation of this paper. He is much indebted to Professor J. G. Truxal, Head of Department of Electrical Engineering, Polytechnic Institute of Brooklyn, for presenting this paper in Moscow on behalf of the author.

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READER OPINION

ON ELECTRIFICATION OF CARBON BLACK

I have read the article on "Electrification of Carbon Black by Contact with a Metal Surface," by Brasefield, which appeared in the October 1960 issue of the JOURNAL. Not enough work has been done in this field and it was a pleasure to learn of these studies.

The electrical charge on cigarette smoke is a somewhat elusive thing to prove and I am curious to know more of this technique. I find it difficult to reach a conclusion regarding the electric charge especially if the only measurements made are those described in the article. There are questions on aging of the smoke particles and distinguishing between charges on smoke and gaseous ions.

In work performed at Stanford University in the Electrical Engineering Department an effort was made to perfect the use of a radioactive probe in a Faraday cage. The results were quite disappointing. Several probes were investigated including polonium, tritium and radium D. Part of the difficulty can be attributed to the great variability of the contaminates in the air and their distance from the radiation source in the space being tested. The polonium, alpha source, gave different results from the hard beta of radium D and different again from the soft beta of tritium. All three cause ionization of air, but because air is such a variable mixture, the space charge that is measured is greatly altered by the difference in ion mobility of charged submicroscopic particles.

In my work with cigarette smoke I came to an opposite view about the electric charge after observing both the effect reported in the article and the direct effect of collecting smoke particles. The probe indicated a negative charge more often than positive but not consistently. Is it possible that the small positive ions become attached to cigarette smoke and the remaining small negative ions are measured by the collecting probe?

Given time to age, the smoke particles acquire a positive charge from the small positive ions—leaving predominantly small negative ions with a high mobility. Since small ions have a mobility more than a thousand times greater than charged cigarette smoke, the charge sensed by the probe connected to the electrometer is negative.

In separate experiments using cigarette smoke and an ion collector designed to measure ions of different mobility it was observed with fresh smoke taken directly through the cigarette, as would occur when a cigarette is smoked by a person, the smoke particles were charged equally positive and negative. The small ions in this set of experiments

were predominantly positive—assumed to be CO_2 and the few negatives, O_2. In a small chamber where the smoke could age for a few seconds the smoke acquired a predominantly positive charge. The ion collector in these experiments was built with two stages and all measurements were made by recording positive and negative ion mobilities simultaneously.

Differences were observed between smoke drawn through the cigarette and smoke taken from the lighted end. Also, smoke taken into the mouth and exhaled was relatively free of charge—showing a high adsorption of charge in the mouth and nasal passages.

In summary, I question whether a Faraday cage test is valid for determining the electrical charge on cigarette smoke. A new instrument being developed at Stanford University may be the solution to the conflicting reports on particle charge and space charge. This instrument is designed to take the complete mobility spectrum of ions in air and automatically record the results. It offers an opportunity to identify some airborne submicroscopic contaminates by the resulting spectrum. A report was made on this new device before the Air Pollution Symposium in New Orleans last March. However, the work is not yet complete and there are technical problems to be solved.

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Author's Reply.

In reply to the above comments, may I first call attention to a report by B. Vonnegut and C. B. Moore entitled "A Study of Techniques for Measuring the Concentration of Space Charge in the Lower Atmosphere," published 31 January 1958 by Arthur D. Little, Inc., Cambridge, Mass., for the Geophysics Research Directorate of the Air Force Cambridge Research Center. The first part of this report, and particularly page 48, presents in considerable detail the theory and use of a Faraday cage for measurement of space charge. It is possible that this presentation will clarify some of the questions raised.

If the radioactive probe in the Faraday cage is at a positive potential with respect to its environment in a cloud of negatively charged smoke, negative ions produced by the alpha particles will be collected by the probe until its potential is lowered to that of its environment. The alpha particles, of course, produce as many positive ions as negatives, and these positive ions will effectively neutralize part of the negatively charged smoke. The effect of these positive ions on the initial electrometer reading will be small if the capacity of the electrometer is small. It is not surprising that results were different depending on whether the radioactive probe was an alpha emitter or a beta emitter, since the space potential measured by the probe is more localized in the

former case. However, the space charge should not be appreciably *altered* by the various probes if the electrometer capacity is small.

We observed, as did Mr. Beckett, that when cigarette smoke is inhaled and held in the lungs for a number of seconds, the charge seems to dissipate because when the smoke is then exhaled into the Faraday cage, the electrometer shows only a small deflection. If a mouthful of smoke is discharged immediately into the Faraday cage, the electrometer generally shows a large negative deflection. However, if air is *blown* through the cigarette in such a way that the smoke from the lighted end drifts into the Faraday cage, we found that the smoke was occasionally positively charged.

We performed another interesting experiment which I shall describe briefly. When it was first observed that cigarette smoke inside the Faraday cage was charged, the question arose as to whether the charge separation took place at the cigarette or at the walls of the Faraday cage as the smoke passed into the cage. We therefore constructed a second cage of the same dimensions as the first, but the walls of the second cage were made of 2-in. chicken wire instead of window screening. The cages were set up near each other and smoke was blown first into one and then into the other. Within the experimental error of these experiments, there was no difference in the electrometer readings of the two cages, suggesting that the smoke was already charged as it approached the cage.

The Faraday cage with polonium probe may be open to criticism as a quantitative instrument for measuring space charge on cigarette smoke but I can see no objection to its use as a qualitative instrument in experiments such as those we have been discussing. Nevertheless, I will be interested in learning more about the instrument under development at Stanford University.

CHARLES J. BRASEFIELD

BOOK REVIEW

OUTER SPACE PHOTOGRAPHY FOR THE AMATEUR, by Henry E. Paul. 124 pages, illustrations, 4¼ × 8 in. New York, Amphoto, 1960. Price, $2.50.

With the dawn of the space age there also was evident a rebirth of interest in celestial objects and astronomical phenomena. This interest was stimulated by the attempts of many lay people to photograph the Sputniks first and then the American satellites that swarmed into the sky. As a result of this new development, astronomers in general and the staff of The Franklin Institute Astronomy Department, in particular, were deluged with calls requesting information on cameras, films, focal lengths, exposure times and other photographic terms in addition to the basic questions in astronomy. Thus, the reviewer and his colleagues found themselves in the unenviable position of trying to answer and interpret technical questions to a lay audience. It could be done but it took considerable time which was unavailable. It was at that time that we wished for an easily understandable book on the photography of celestial objects.

In *Outer Space Photography for the Amateur*, my very good friend Dr. Henry E. Paul has written a book with authority and balance. In an inexpensive, small book, the author has explored the entire subject of celestial photography. It is written in a lucid, non-technical fashion to permit easy reference and provide the essential information to permit the photographing of any object in the sky.

As an example of the content, we find that for satellites we need a fast lens F:4.5 or faster with a 40 to 60 degree field and the very fastest film. You just open the shutter and record the trace of the satellite.

Do you want to photograph the moon? You need a long focal length lens or the image is too small to use. Also, you need a yellow filter to correct for the color-error in the lens. Dr. Paul recommends as a starting point you photograph the quarter moon at F:8 at 1/50th of a second on Panatonic-X film.

The sun is a more difficult subject and requires extensive experimentation with a series of filters. Using a 100,000× filter, the author recommends 1/60th of a second at F:11 on the slowest film available.

As far as stars are concerned, you can get snapshots of these. Even color photographs are possible. What about a drive for the telescope? The author shows that the motion of the star image in one second of time is 0.0007X focal length in inches. Comets, Meteors, Nebulae, Aurorae are all covered in this work.

The author is a recognized master in the design, manufacture and use of optical equipment. His versatility and knowledge have been distilled and are contained between the covers of his superlative new book. I will personally recommend this book to the many questioners who desire the answers to problems in astronomical photography. At twice the modest cost of $2.50 this would be a striking bargain in books. At this price, no one with a telescope or camera can afford to be without it.

I. M. LEVITT
The Fels Planetarium

BOOK NOTES

THE CONTROL OF MULTIVARIABLE SYSTEMS, by Mihajlo D. Mesarović. 112 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc.; Cambridge, The Technology Press; 1960. Price, $3.50.

This significant contribution to the theory of control systems is divided into two parts: Part I covers interrelations of multivariable systems and Part II is on the synthesis of such systems. The author's thesis is that control systems should have a multivariable foundation rather than a single-variable one. He developed this theory in 1958–1959 while he was a Sloan Foreign Post Doctoral Fellow at MIT. He uses a V-canonical structure with a feedback controller as the model for his multivariable theory and outlines situations in which combinations of the P, V, and H structures are suitable. Readers of the JOURNAL will recall Dr. Mesarović's paper in the April 1960 issue, on "Dynamic Response of Large Complex Numbers." A point of interest to the JOURNAL is that the list of references includes 33 items, 4 of which are JOURNAL papers, 15 are books, 5 are reports or unpublished papers, 8 are from regularly published society transactions, and 1 is from a Russian journal.

SPACE TRAJECTORIES, edited by the Technical Staff of the Research Division of Radiation Incorporated. 298 pages, diagrams, 6 × 9 in. New York, Academic Press Inc., 1960. Price, $12.00.

This volume is the published proceedings of a two-day symposium held in December 1959, to survey current space trajectory analysis. Sponsors of the symposium were the Advanced Research Projects Agency, The American Astronautical Society, and the Research Division of Radiation Incorporated. The fifteen papers comprising the volume deal with military space, satellite and probe orbit computation, trajectory constants, trajectory computation, computer programs for space missions, space-borne computer design, re-entry trajectories, and space-trajectory instrumentation. The fifteenth paper on computer programming methods is a composite of six different programs. Discussion follows some of the papers; some have up-to-date lists of references. This is a valuable reference work for the fast-growing science of space exploration.

CONTRIBUTIONS TO THE THEORY OF NONLINEAR OSCILLATIONS, edited by L. Cesari, J. LaSalle and S. Lefschetz. 286 pages, diagrams, 7 × 10 in. Princeton, Princeton University Press, 1960. Price, $5.00 (paper).

Number 45 in the Princeton Annals of Mathematics Studies, this is Vol. V of the "Contributions to the Theory of Nonlinear Oscillations." The variety of topics covered in Vol. V includes optimal control, Poincaré's Perturbation Method, fixed point theorem, non-descriptive algebra, Lagrange stable motions, asymptotic stability in 3-space, and periodic solutions of various systems. Twelve of the thirteen papers are in English; the other is in French (on a system of trajectories of dynamic systems).

INTRODUCTION TO QUANTUM MECHANICS, by R. H. Dicke and J. P. Wittke. 369 pages, diagrams, .6 × 9 in. Reading (Mass.), Addison-Wesley Publishing Co., Inc., 1960. Price, $8.75.

This text is designed for an introductory course in nonrelativistic quantum mechanics at the graduate level, but it may be used for advanced undergraduates. Prerequisites are a knowledge of calculus and differential equations and a course of basic physics including classical mechanics. The final chapter deals with quantum-statistical mechanics, a subject previously neglected in texts on quantum theory.

MOLECULAR DISTILLATION, by G. Burrows. 211 pages, diagrams, 5½ × 8½ in. New York, Oxford University Press, 1960. Price, $5.60.

This book is primarily intended for the laboratory research worker. The underlying principles are dealt with according to simple kinetic theory and also by means of the standard terminology used for distillation processes. Engineering problems associated with the production and maintenance of the

necessary degree of vacuum are given attention. Gas and vapour flow at low pressures also receive detailed treatment. Examples of degassers and molecular stills are described together with typical applications.

SURFACE MICROTOPOGRAPHY, by S. Tolansky. 296 pages, illustrations, 6 × 8¼ in. New York, Interscience Publishers, Inc., 1960. Price, $9.00.

This work aims to illustrate the wide field open for study of this subject and to present a good deal of original matter. It wishes to show what can be achieved by precision multiple-beam interferometry when applied to the microtopographies of surfaces. It is a full presentation of the theory and practice of those interferometric techniques that the author himself has developed. Some 350 new interferograms are given, which demonstrate a powerful three dimensional microscopy; in the up-down direction magnifications of ×500,000 and resolutions of even quite small crystal lattice spacings have been achieved, yet by very simple means. The book's many topics include microtopographical studies on both natural and cleavage faces of a variety of crystals; the etching of crystals; studies of oscillating quartz crystals; the finish of machined and polished surfaces; the study of hardness test indentations; erosion effects produced by high speed impact; and erosion due to electric sparks. It should be of special interest to industrial and academic physicists, crystallographers, mineralogists, engineers, chemists and metallurgists, to electro-technics specialists and to the diamond, machine tool and watch industries.

POLYPROPYLENE, by Theodore O. J. Kresser. 268 pages, illustrations, 5 × 7¼ in. New York, Reinhold Publishing Corp., 1960. Price, $6.50.

All phases of polypropylene properties, production and applications are brought together in this concise handbook. It offers information on this increasingly important thermoplastic at a level where it can be readily understood by readers without special training in polymer chemistry. Particular emphasis has been placed throughout the book on the relationship between the development of polypropylene and the future of high polymers in general. The author has also included sufficient information on the basic principles of stereochemistry, to point up the unique qualities of polypropylene. Included as a special feature is a chapter on chemistry that describes the work of Professor Guillo Natta on the discovery and development of polypropylene.

PRINCIPLES OF FEEDBACK CONTROL, by Charles H. Wilts. 271 pages, diagrams, 6 × 9 in. Reading (Mass.), Addison-Wesley Publishing Co., Inc., 1960. Price, $8.75.

This treatment of the analytical methods used in the design of feedback systems is intended for use as a textbook in courses at the advanced undergraduate-graduate level, and for practicing engineers who wish to study the principles of feedback control. The reader is presumed to have a good background in ordinary differential equations and complex variable theory. Although desirable, the student is not required to have any formal training in the use of the Laplace transform for solving linear differential equations. The main theme of the book is the stability problem. Attention is focused on the fundamental aspects of the problem, to the Routh-Hurwitz and Nyquist tests, to the generalized concept of a root locus, and finally to the problem of creating a satisfactory degree of relative stability in systems that are either unstable or insufficiently stable.

ON THERMONUCLEAR WAR, by Herman Kahn. 651 pages, 6 × 9¼ in. Princeton, Princeton University Press, 1960. Price, $10.00.

The author, whose "briefings" on the horrors of total war have served to stir up military and civilian controversy, prepared this volume while he was a Visiting Research Associate at Princeton. In his preface, Mr. Kahn says "This book examines the military side of what may be the major problem that faces civilization, comparing some of the alternatives that seem available and some of the implications in these choices." He has restricted the discussion to the deterrence and waging of all-out thermonuclear wars between the United States and the Soviet Union. He dedicates his book to "the goal of anticipating, avoiding and allevi-

ating crises." There are twelve chapters and five appendices, covering such topics as genetic consequences, the SAC, problems of arms control, nonmilitary and military deterrents, Civil Defense programs, timing and coordination problems. In tables of data and through dispassionate analysis, Mr. Kahn makes it only too clear that unless we understand that the problem is one of sheer survival and are willing to plan for and spend enough money adequately to prepare for all eventualities, we cannot hope to avoid catastrophe. Mr. Kahn's main purpose in writing the book is to show that it is possible to decrease the probability of catastrophic nuclear war and to lessen its destructive effects if it does occur. The book should be read by all Congressmen, state officials, and those who are responsible for formulating our national defense policy. All readers will not agree with Mr. Kahn's analysis, but they should find it difficult to refute his objective arguments.

INTRODUCTION TO MECHANICS, MATTER AND WAVES, by Uno Ingard and William L. Kraushaar. 672 pages, diagrams, 6 × 9 in. Reading (Mass.), Addison-Wesley Publishing Co., Inc., 1960. Price, $9.75.

This elementary textbook, intended for use in the first year of a two-year course in general physics for students of science and engineering, is based on such a course taught at MIT by the authors. Basic concepts of mass, momentum, energy, angular momentum and their conservation laws are stressed. The authors' aim has been to maintain a balance between experiment and inductive reasoning. Sample chapter titles are: Inertial Mass, Impulse and Work, Examples of Forces and Motion, Temperature and Heat, Atoms and Molecules, Some Properties of Matter, Mechanics of Fluids, and Transverse and Longitudinal Wave Pulses. Numerous problems are included; answers are provided for the odd-numbered ones.

THE CHEMISTRY AND TECHNOLOGY OF FERTILIZERS, edited by Vincent Sauchelli. ACS Monograph 148. 692 pages, illustrations, 6 × 9 in. New York, Reinhold Publishing Corp., 1960. Price, $18.00.

This monograph covers the major changes and developments in the chemistry and technology of the fertilizer industry since 1935. In 24 chapters contributed by experts from government, industry and agriculture, the book covers raw materials, manufacture, processing and use of chemical fertilizers. Special chapters are devoted to the caking of mixed fertilizers, materials-handling equipment and gaseous effluents. The editor, in his foreword, states that chemical fertilizers, although relatively a new agricultural development, "are one of the most strategic weapons of modern agriculture." The monograph brings up-to-date the information on this strategic weapon and suggests possible progress for the future.

CONTINUOUS GEOMETRY, by John von Neumann. 299 pages, 6 × 9 in. Princeton, Princeton University Press, 1960. Price, $7.50.

Continuous geometry was invented by John von Neumann in 1935. It is a remarkable extension of projective geometry to the non-finite dimensional case, just as Hilbert and Banach spaces are such an extension for vector spaces. Now published posthumously, these notes represent lectures given in 1935 and 1937 at Princeton University, and at the Institute for Advanced Study. The text is divided into three parts. Part I develops the axioms of continuous geometry, dimension theory and for irreducible case the dimension function $D(a)$. Part II presents von Neumann's discovery that every complemented modular lattice of order ≥ 4 can be represented as the lattice of all principal right ideals of a suitable ring. Part III begins the dimension theory for the reducible case of continuous geometry and gives the fundamental decomposition techniques, the notion of central cover and the unrestricted transitivity of perspectivity.

FROM DUALISM TO UNITY IN QUANTUM PHYSICS, by Alfred Landé. 114 pages, 5¼ × 8¼ in. New York, Cambridge University Press, 1960. Price, $3.75.

The author, in challenging the view that the principles of wave-particle duality, complementarity and uncertainty are fundamental to micro-mechanics, sets out to reduce the dualistic quantum phenomena to still more elementary general principles of sym-

metry, invariance, continuity and the like. He thus develops the current Bohr-Heisenberg account, resting on Born's statistical interpretation, into a unitary theory. The book begins with a philosophical discussion of determinism and chance, in which the uncertainty of an individual event in a statistical group is linked to the postulate of macro-physical cause-effect continuity. Chapter Two deals with the states of a micro-physical object in general, and the transition probabilities from state to state in reaction to macro-physical measuring instruments. Chapter Three suggests that the quantum law of probability interference is not a mere oddity, but a consequence of the nature of probability itself as a lawful connection between events. Chapter Four gives a deduction of the most characteristic quantum feature—the wave-like relation between co-ordinates and momenta—from non-quantal theorems. The book concludes with a critical survey of present fashions in quantum philosophy and language.

AN INTRODUCTION TO HOMOLOGICAL ALGEBRA, by D. G. Northcott. 282 pages, diagrams, 6 × 9 in. New York, Cambridge University Press, 1960. Price, $8.00.

This new mathematical subject has emerged during the past ten years. Because of its fundamental nature it is relevant to many branches of pure mathematics, including number theory, geometry, group theory and ring theory, and it is a subject which mathematicians wishing to keep informed about major developments, cannot neglect. The author aims to introduce homological ideas and methods, and to show some of the results which can be achieved. The early chapters provide the results needed to establish the theory of derived functors, and to introduce torsion and extension functors. The new concepts are then applied to the theory of global dimension, in an elucidation of the structure of commutative Noetherian rings of finite global dimension, and in an account of the homology and cohomology theories of monoids and groups. A final section is devoted to comments on the various chapters, supplementary notes, and suggestions for further reading. This work will be useful as a reference for specialists as well as a helpful and lucid account for those about to begin research.

DIGITAL APPLICATIONS OF MAGNETIC DEVICES, edited by Albert J. Meyerhoff. 604 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1960. Price, $14.00.

The first comprehensive text on digital magnetic devices, this volume will be a boon to students and professional computer engineers. It is based on the research programs sponsored by Burroughs Corporation on magnetic-core devices and magnetic-shift registers. The book covers theory, electrical design and logical design necessary for the development of digital systems using the magnetic core as the basic circuit element. The text is divided into the following parts: Fundamentals, Parallel Magnetic Pulse Amplifiers, Delay Parallel Magnetic Pulse Amplifiers, Series Magnetic Pulse Amplifiers, Memories, Transistor-Magnetic Pulse Amplifiers, and Other Techniques. A good feature is the inclusion of a list of symbols at the beginning of each part.

ANNUAL REPORTS ON THE PROGRESS OF CHEMISTRY FOR 1959. Vol. LVI, 476 pages, diagrams, 5½ × 8½ in. London, The Chemical Society, 1960. Price, 2 pounds, post free.

The 1959 volume of these Annual Reports is divided into five sections: General and Physical Chemistry (9 papers); Inorganic Chemistry (3 papers); Organic Chemistry (11 papers); Biological Chemistry (5 papers); and Analytical Chemistry (7 papers). The several papers fall into the same general classifications as last year's volume, but they either cover different phases of the subject, or extend the coverage. One entirely new topic (for the Reports) occurs in the Biological section—the chemistry of bone and tooth minerals. As in past volumes, the reference lists are quite valuable, covering for the most part only 1959 entries. References are given as footnotes throughout the text; an alphabetical index of authors referred to is included at the end of the volume.

THE STORY OF STEREO: 1881, by John Sunier. 160 pages, illustrations, 5½ × 8½ in. New

York, Gernsback Library, Inc., 1960. Price, $5.00 (hard cover); $2.95 (soft cover).

The history and development of stereo from its earliest beginnings in the last century right up to the present are given in this book. It is also designed to aid in an understanding of the principles behind stereophony and what the medium has accomplished. The applications of stereo on film, tapes and discs, and also in broadcasting have been covered by the author. He concludes by describing stereo techniques in the home, in business and industry and in medicine.

ELECTRONIC MAINTAINABILITY, edited by F. L. Ankenbrandt. Vol. 3. 312 pages, illustrations, 6 × 9 in. Elizabeth (N. J.), Engineering Publishers, 1960. Price, $10.00.

This book is based on the Third Electronic Industries Association Conference on Maintainability of Electronic Equipment, held in San Antonio, Texas. Twenty-nine papers are herein presented giving an authoritative discussion of the developments which have been changing maintainability from the vaguely practiced art of yesterday to the rigorous science of today and tomorrow. Subject matter ranges from maintenance problems of space flight to the determination of adequate working space for electronic technicians. The contributors are among the foremost experts in the field. Although concerned primarily with military electronic equipment, much of the material is equally applicable to more prosaic products such as automobiles, household appliances, plumbing fixtures, and what have you.

OFF-THE-ROAD LOCOMOTION, by M. G. Bekker. 220 pages, illustrations, 6 × 9 in. Ann Arbor, University of Michigan Press, 1960. Price, $10.00.

Mr. Bekker, whose interesting paper in this JOURNAL for May 1957 introduced many of our readers to his work in off-the-road locomotion, presents a comprehensive treatment of this new approach to an age-old transportation problem. Mr. Bekker's main contribution to terramechanics is to relate the vehicle as a whole to the terrain it must operate on. Methods of predicting vehicle performance, soil and snow mechanics, tracks, wheels and tires are all treated in this compact volume, with emphasis on the fact that sound theory must be used in designing equipment in these days when "cut-and-try" methods are too slow and expensive. A wealth of graphs and photographs is most helpful, especially when the author delves into material not familiar to the reader. In the chapter describing the effective spaced link track, however, the eight photographs of Fig. 75 have been reduced so greatly in reproduction that all detail has been lost in a meaningless blur.

CLASSICAL ELECTRICITY AND MAGNETISM, by E. S. Shire. 396 pages, diagrams, 5½ × 8½ in. New York, Cambridge University Press, 1960. Price, $7.50.

This volume based on classical electricity and magnetism bridges the gap between the elementary account as taught in schools and a mathematical treatment based on Maxwell's equations. It deals with the "classical," that is, macroscopic, aspects of the subject to the standard of a general degree course in physics or engineering or of the first part of an honors degree course. The author combines an experimental with an historical approach. There are chapters on the measurement of electrical quantities and on applied electricity, the latter including introductory accounts of electronics and of machines for accelerating charged particles. Simple vector methods are used but vector calculus equivalents of the more important equations are confined to footnotes and appendices. Answers are given to the wide range of examples. In the belief that a physicist should be able to adopt whichever system of units is most convenient for the purpose in hand, the author has not tied his book to any one system, but has made it equally applicable to "rationalized" and "unrationalized" m.k.s. and c.g.s. systems. The symbols used are those recommended by the International Union of Pure and Applied Physics.

PUBLICATIONS RECEIVED

LAPLACE TRANSFORMATION, by William T. Thomson. Second edition, 255 pages, diagrams, 6 × 9 in. Englewood Cliffs, Prentice-Hall, Inc., 1960. Price: $10.00 (trade); $7.50 (text).

TUBES AND CIRCUITS, by George J. Christ. 192 pages, diagrams, 5½ × 8½ in. New York, Gernsback Library, Inc., 1960. Price: $5.00 (hard cover); $3.45 (paper).

MECHANICAL-ELECTRICAL EQUIPMENT HANDBOOK FOR SCHOOL BUILDINGS, by Harry Terry. 412 pages, illustrations, 6 × 9 in. New York, John Wiley & Sons, Inc., 1960. Price, $9.50.

ACTIVATION ANALYSIS HANDBOOK, edited by R. C. Koch. 219 pages, 8 × 10½ in. New York, Academic Press Inc., 1960. Price, $8.00.

FUNDS FOR RESEARCH AND DEVELOPMENT IN INDUSTRY, 1957. 119 pages, diagrams, 8 × 10 in. National Science Foundation Survey 60–49. Washington, Government Printing Office, 1960. Price, $0.65 (paper).

MATHEMATICAL TABLES. VOLUME 3. TABLES OF GENERALIZED EXPONENTIAL INTEGRALS, by G. F. Miller. 43 pages, 8½ × 10¾ in. London, Her Majesty's Stationery Office, 1960. Price, 7s, 6d. American agents, British Information Services in New York; price, $1.43 postpaid.

SCIENTIFIC AND TECHNICAL PERSONNEL IN AMERICAN INDUSTRY. Report on a 1959 Survey. 66 pages, 8 × 10½ in. National Science Foundation Survey 60–62. Washington, Government Printing Office, 1960. Price, $0.45 (paper).

LES STRUCTURES DE COMMUTATION A m VALEURS ET LES CALCULATRICES NUMERIQUES, by Midhat J. Gazalé. 78 pages, 6½ × 9½ in. Paris, Gauthier-Villars, 1959. Price, $3.04 (paper).

NATIONAL BUREAU OF STANDARDS NEWS

THE AMOS IV COMPUTER

The National Bureau of Standards in cooperation with the U. S. Weather Bureau has developed a specialized digital computer[1] for the Weather Bureau to use as a research tool in exploring the concept of the automatic weather station. The AMOS IV computer receives data from weather-sensing instruments and processes these data through such functions as sampling, comparing, selecting a maximum, and arithmetic operations. The results are transmitted via teletype to a central forecasting station and to other airport weather stations. Values of two quantities recently developed as aids to air safety—runway visual range and approach light contact height—are given by the machine through automatic table look-up.

For a number of years, the Weather Bureau has been appraising the possibilities of an automatic weather station. Such stations could be widely distributed, and would be especially useful in relatively inaccessible locations that are important sources of early data on meteorological activity. The various developmental prototypes of this concept have been called AMOS (Automatic Meteorological Observation Station); the current version, containing transistorized packages, is AMOS IV. This model was designed and built by Paul Meissner and J. A. Cunningham of the NBS data processing systems laboratory and by C. A. Kettering of the U. S. Weather Bureau. It is an outgrowth of previous work done by NBS for the Weather Bureau that resulted in a special computer[2] for processing cloud-height signals from a ceilometer. The ceilometer was intended for use with the AMOS III.

Several of the input quantities to the AMOS computers, such as cloud height and precipitation, cannot be satisfactorily represented by instantaneous values but must be time-averaged. Varying amounts of data processing must therefore be associated with the different instruments measuring these quantities. In the AMOS III concept, several complex units were required for these functions. Although many of the functions were similar, the hardware was not minimized because of a diversity of design that resulted from the isolated development of the individual units. Analysis of the over-all system indicated that a considerable reduction could be made in hardware and therefore in maintenance.

In AMOS IV, the automatic weather station is built around a single small, general-purpose computer designed especially for this application.

[1] "A Computer for Weather Data Acquisition," by P. Meissner, J. Cunningham, and C. Kettering (to be published *Proc. of EJCP*, 1960).

[2] "Cloud-Height Data Analyzer," *NBS Tech. News Bull.*, Vol. 43, p. 180 (1959).

The computer receives data from the input instruments at any desired interval. These data are suitably processed and arranged in a specified order for teletype transmission in a variety of message formats and at various speeds. The computer also operates local and remote displays. Much latitude is available for research into the most desirable form of data processing because of the inherent flexibility of the internally programmed machine.

The computer continuously monitors new input data while simultaneously processing data already entered and transmitting messages on command. Among the input quantities which the AMOS IV computer can handle are temperature, dew point, wind speed and direction, atmospheric pressure, precipitation, transmissivity, and cloud height. Input data can be received directly from the instruments in the simplest possible form, such as analog voltage, current, or resistance; and pulse rate or contact closure. Information may also be received in coded form, such as the Gray binary code frequently used with shaft-position encoders. The nature of the weather instruments and of the quantities measured limits the input data to 2 or 3 decimal digits for the most part; word size is therefore 3 digits plus sign. Double-precision methods are available for those few instances requiring greater accuracy. Communication with the machine is via electric typewriter or punched tape.

The computer circuitry is based on transistorized plug-in assemblies[3] designed at NBS for a variety of data-processing applications. These 50-kc packages perform flip-flop, analog switch, and gating circuitry functions, as well as others.

To store data, the machine uses a magnetic drum operating at 1800 rpm that carries 100 general storage channels of 100 words each and has space for 100 additional channels. Several dual-head channels are available for simultaneous read-in and read-out of incoming data, outgoing messages, etc. The magnetic drum provides the extensive storage capacity required for the table look-up involved in the calculations of runway visual range and approach light contact height. About 35 tables are stored on the drum; each table has about 90 three-digit values.

One set of these tables contains the data on runway visual range (RVR), i.e., the distance along the runway visible to a pilot from the point of touchdown—generally 1000 to 6500 feet, depending upon runway illumination (natural and artificial) and atmospheric conditions. The primary input for the RVR determination is a transmissometer reading. The computer continuously monitors this reading and "looks up" the proper corresponding value of RVR, which is then displayed locally and inserted into the teletype message.

The other set of tables contains the data on approach light contact height (ALCH), i.e., the height from which the pilot can identify the

[3] "Packaged Switching Circuits," *NBS Tech. News Bull.*, Vol. 43, p. 184 (1959).

approach lights. ALCH is affected by background illumination level, atmospheric conditions, and the intensity of the approach lights, which are set in accordance with prevailing conditions. If limiting conditions are indicated by either low clouds, as shown by the ceilometer, or by fog or snow, as sensed by the transmissometer, a value of ALCH based on the interfering factor is obtained. If both factors are present, two calculations are made; the machine then determines and displays the lower value. Since there is a statistical uncertainty in this type of information, two values of altitude are presented. The higher altitude is that at which the pilot has a 20 percent probability of seeing the approach lights; the lower altitude is that at which the probability is 90 percent.

YOU CAN ADVANCE SCIENCE EDUCATION

Today, more than ever before in its 135-year history, there is vital need for The Franklin Institute to effectively promote education in science and technology. It is imperative that we meet this challenge by providing adequate educational opportunities in these fields. This requires vision, objective planning, and money. We have more than enough of the first two requisites, but far too little of the third.

Our programs are aimed at professional scientists and industry, as well as the lay public and young people seeking inspiration and guidance in choosing a career. The Institute's educational programs are impressive, for they begin with students in the early grades of our elementary schools and continue throughout an individual's professional or industrial life. With more funds at our disposal, the scope and vigor of these activities could be greatly increased.

The Franklin Institute is not richly endowed. It is a non-profit organization, depending for encouragement and support on an understanding public. Capable and conservative management assures wise administration of all funds.

Your gift or bequest, large or small, will be deeply appreciated and will be used effectively to broaden the Institute's educational usefulness. There is a warm satisfaction in giving financial support to an organization that has pioneered in, and is dedicated to, the advancement of science and technology through education.

When property is transferred, title should be in the name of The Franklin Institute of the State of Pennsylvania for the Promotion of the Mechanic Arts.

The Secretary of The Franklin Institute will gladly furnish you with additional information. Write to him at The Franklin Institute, Benjamin Franklin Parkway at Twentieth Street, Philadelphia 3, Pennsylvania.

THE FRANKLIN INSTITUTE

MINUTES OF THE ANNUAL MEETING

January 18, 1961

The Annual Meeting of The Franklin Institute was held at 8:15 P.M. in the Lecture Hall. Wynn Laurence LePage, President, called the meeting to order with approximately 250 members and guests in attendance.

The President stated the minutes of the Stated Meeting of November 18, 1960 were published in the December issue of the JOURNAL. There being no additions or corrections, the minutes were approved as published. The minutes of the Stated Meeting of December 21, 1960, he stated, will be published in the January issue of the JOURNAL and will be presented for approval at the next Stated Meeting.

The President then called on the Chairman of the Tellers of Election, Dr. Joseph S. Hepburn, for the Report of the Tellers who had been appointed in accordance with the By Laws to count the votes for ten Managers to be elected. Dr. Hepburn reported that 2891 votes were cast, of which 2502 were legal and 389 defective, and the nominees received votes as follows:

Morton Gibbons-Neff	2037	Wynn Laurence LePage	2325
Rupen Eksergian	2192	Gaylord P. Harnwell	2332
Francis J. Chesterman	2290	Emery W. Loomis	2254
James Creese	2311	Henry M. Chance, II	2263
Henry B. Bryans	2253	Joseph A. Fisher	2264
		Kenneth Robert Atkins	1507

The President then declared the following ten members were thereby elected Managers of The Franklin Institute for a term of three years:

Morton Gibbons-Neff	Wynn Laurence LePage
Rupen Eksergian	Gaylord P. Harnwell
Francis J. Chesterman	Emery W. Loomis
James Creese	Henry M. Chance, II
Henry B. Bryans	Joseph A. Fisher

The President thanked Dr. Hepburn, Mr. Duerig and Mr. Neher for their services as Tellers of Election.

Mr. LePage then called on the Secretary of The Institute to read the report submitted by the Trustees of the Elliott Cresson Medal Fund showing the financial transactions for the year, which, in accordance with the Deed of Gift that forms the basis of the Award, must be presented at the Annual Meeting of The Franklin Institute. The report follows:

ELLIOTT CRESSON MEDAL FUND

Report of the Trustees to The Franklin Institute
for year ended December 31, 1960

PRINCIPAL

Principal of the Fund (Representing the original $1,000 plus transfers thereto from Unexpended Income and profit from disposition of investments)		$4,304.73
Invested as follows:		
$4,000 U. S. Treasury 2½% bonds, due 6/15/1962–67 at book value	$4,023.79	
Cash in Western Savings Fund Society	280.94	
	$4,304.73	

INCOME

Income of the Fund for 1960		
Interest on bonds (less amortization)	$ 121.44	
Interest on bank balance	9.14	$ 130.58

Expenses		
Medals, certificates, insurance and cases		232.89
Expenses in excess of Income for the year 1960		(102.31)
Proceeds from Sale of $1,000 U. S. Treasury Bond		936.25
		$ 833.94
Unexpended Income at January 1, 1960	1,971.49	
Less: Value of bond sold	1,005.98	965.51
Unexpended Income at December 31, 1960		$1,799.45
Invested as follows:		
$1,000 U. S. Treasury 2½% bonds, due 6/15/1962–67 at book value	1,005.98	
Cash in Fidelity-Philadelphia Trust Company	793.47	
	$1,799.45	

/s/ *Henry B. Allen*
HENRY B. ALLEN

/s/ *Delmer S. Fahrney*
DELMER S. FAHRNEY
Trustees

January 18, 1961

The President presented The President's Letter, which will accompany the Annual Report of The Franklin Institute for 1960, which report will be distributed to the membership of The Institute early this year. The President's Letter contained highlights covering The Institute's activities for 1960.

Mr. LePage then presented Sir George Thomson, Master of Corpus Christi, Cambridge, and President of the British Association for 1960, as the speaker of the evening. Sir George spoke on "The Importance of Useless Science." His talk on science as one of the great achievements of the human intellect emphasized the debt our technology owes to free investigation and was illustrated with interesting slides.

After thanking Sir George for his challenging talk, and before calling for questions, Mr. LePage presented Sir George with a copy, from our own Library, of his father's book, titled, "The Electron in Chemistry" which covers a series of five lectures delivered by Sir Joseph J. Thomson before The Institute in 1923. In this book, Mr. LePage said, he had taken the liberty of inscribing, "To Sir George Thomson in memory of your late father, Sir Joseph J. Thomson, whose scientific career was revolutionary in its successes," and had signed as President of The Franklin Institute.

Following an interesting question period, the President adjourned the meeting at 9:45 P.M.

WILLIAM F. JACKSON, JR.
Secretary

NEW BOOKS IN THE FRANKLIN INSTITUTE LIBRARY

MATHEMATICS

STATISTICAL THEORY OF COMMUNICATION, by Y. W. Lee. Wiley, 1960.

AN INTRODUCTION TO THE THEORY OF NUMBERS, by I. M. Niven. Wiley, 1960.

AN INTRODUCTION TO THE THEORY OF EXPERIMENTAL DESIGN, by D. J. Finney. University of Chicago Press, 1960.

AUTOMATIC TRANSLATION, by D. I. Panov. Pergamon, 1960.

ASTRONOMY

LAROUSSE ENCYCLOPEDIA OF ASTRONOMY, by L. Rudaux. Batchworth Press, 1959.

MOON MAPS, by H. P. Wilkins. Macmillan, 1960.

THE PLANETARY EQUATORIUM OF JAMSHID GHIYATH AL-DIN AL-KASHI, with translation and commentary by E. S. Kennedy. Princeton University Press, 1960.

INTRODUCTION TO THE MECHANICS OF THE SOLAR SYSTEM, by R. Kurth. Pergamon, 1959.

SOURCE BOOK IN ASTRONOMY, edited by H. Shapley. Harvard University Press, 1960.

PHYSICS

LA THÉORIE DES GAZ NEUTRES ET IONISÉS. Wiley, 1960.
ENGINEERING THERMODYNAMICS, by N. A. Hall. Prentice-Hall, 1960.
THEORETICAL ELASTICITY AND PLASTICITY FOR ENGINEERS, by D. E. R. Godfrey. Thames and Hudson, 1959.
SEMICONDUCTORS, by R. A. Smith. Cambridge University Press, 1959.
PLASMA PHYSICS, by S. Chandrasekhar. University of Chicago Press, 1960.
SPECIAL RELATIVITY, by W. Rindler. Interscience, 1960.
PROPERTIES OF MATTER, by F. C. Champion. 3rd ed. Philosophical Library, 1959.
LECTURES ON THEORETICAL RHEOLOGY, by M. Reiner. 3rd ed. Interscience, 1960.
LOW-TEMPERATURE TECHNIQUES, by F. Din. Interscience, 1960.
CONFERENCE ON SILICON CARBIDE, Boston, 1959. Proceedings, edited by J. R. O'Connor. Pergamon, 1960.

CHEMISTRY AND CHEMICAL ENGINEERING

THE CHEMISTRY OF PROPELLANTS, edited by S. S. Penner. Pergamon, 1960.
BORON FLUORIDE AND ITS COMPOUNDS AS CATALYSTS IN ORGANIC CHEMISTRY, by A. V. Topchiev. Pergamon, 1959.
IMPERFECTIONS IN CRYSTALS, by H. G. van Bueren. Interscience, 1960.
CHEMISTRY AND TECHNOLOGY OF FERTILIZERS, by V. Sauchelli. Reinhold, 1960.
STABILIZATION OF FREE RADICALS AT LOW TEMPERATURES, edited by A. M. Bass. U. S. Dept. of Commerce, National Bureau of Standards, 1960.
SYMPOSIUM ON SPECTROSCOPY. ASTM, 1960.
FOOD CHEMISTRY, by L. Meyer. Reinhold, 1960.
X-RAY POWDER PHOTOGRAPHY IN INORGANIC CHEMISTRY, by R. W. M. D'Eye. Academic Press, 1960.
THEORY OF DETONATION, by I. B. Zel'dovich. Academic Press, 1960.
GROWTH OF CRYSTALS, v. 2, edited by A. V. Shubnikov. Consultants Bureau, 1959.
SELECTED READINGS IN GENERAL CHEMISTRY, by W. F. L. Kieffer. Division of Chemical Education of the American Chemical Society, 1959.

MATERIALS SCIENCE

MATERIALS IN NUCLEAR APPLICATIONS. ASTM, 1960.
SYMPOSIUM ON RADIOISOTOPES IN METALS ANALYSIS AND TESTING. ASTM, 1960.
SYMPOSIUM ON VISUAL AIDS FOR STANDARDIZING AND COMMUNICATING PRODUCT APPEARANCE. ASTM, 1959.
CORROSION PROBLEMS OF THE PETROLEUM INDUSTRY. Macmillan, 1960.
ZONE REFINING AND ALLIED TECHNIQUES, by N. L. Parr. G. Newnes, 1960.
SYMPOSIUM ON HYDRAULIC FLUIDS. ASTM, 1960.

COMMITTEE ON SCIENCE AND THE ARTS

(Abstract of Proceedings of Stated Meeting held Wednesday, January 11, 1961.)

Hall of the Committee,
Philadelphia, January 11, 1961.

MR. J. PHILIP EVANS in the Chair.

The following report was presented for final action:

No. 3325: The Bubble Chamber.

This report recommended the award of an Elliott Cresson Medal to Donald A. Glaser, of Berkeley, California, "In consideration of his invention of the bubble chamber, a device in which the tracks of high energy ionizing particles and of the fragments of nuclear collisions can be photographed, thus providing a powerful tool for obtaining new information about elementary particles and processes that give rise to them."

D. S. FAHRNEY,
Secretary to Committee

CURRENT TOPICS

Hypersonic Shock Tunnel.—Some of the conditions that will be encountered by the first interplanetary space ship as it comes in for a landing on Mars or Venus are being simulated at the General Electric Research Laboratory in Schenectady, N. Y. Velocities many times the speed of sound, in atmospheres believed to be the same as those of Mars and Venus, are being produced in a shock tunnel. By studying the effects of such velocities on various shapes, scientists are getting a preliminary idea of what kind of space ship an astronaut will use in his first planet-hopping expedition.

The General Electric shock tunnel has produced some of the highest flow velocities of air ever reported, up to 35 times the speed of sound, and is the first to provide two separate test chambers of identical high-speed air flow. The provision of two test chambers in a hypersonic shock tunnel permits many different types of measurements to be made under absolutely identical conditions for the first time.

In conventional shock tunnels, only a limited number of measurements can be made during one test run, and it is extremely difficult to duplicate test runs exactly.

The shock tube itself is simply a device for putting a large amount of energy into a quantity of gas. A mixture of hydrogen and oxygen, diluted with varying amounts of helium, is ignited, producing a controlled explosion. This "explosion" ruptures a stainless steel diaphragm separating the burning mixture from low pressure air in the rest of the tube. The shock thus produced travels the 100-ft. length of the tube, and bounces back from the far end. For a few thousandths of a second the air at this end of the tube is compressed to an extremely high pressure and temperature. Two nozzles opening off the end of the tube allow the highly compressed air to expand and flow over small models being tested. Depending on the size and shape of the nozzle opening and the strength of the initial shock, the flow of air over the test models may be as high as 35 times the speed of sound, or Mach 35.

Many tests have been made at flows between Mach 20 and 25, with work now progressing in the range of Mach 25 to 35. Research at these speeds is of importance primarily for space vehicles, since Mach 35 is about the speed needed for escape from the earth's gravitational field. Even high speeds, up to Mach 50 in air, are probably attainable in the General Electric shock tunnel. In addition to high speed, practically any desired atmospheric conditions can be provided, and re-entry into any known atmosphere can be simulated.

Shock tunnel studies provide some of the data necessary to predict how hot a space vehicle will get when re-entering the atmosphere, how the heat is distributed over the surface, and what pressures may be expected. Other types of shock tubes, such as the plasma jet or arc tunnel, are also being used by General Electric to study re-entry problems at its Missiles and Space Vehicle Department in Philadelphia. These studies are providing the basis for choice of materials and shapes for space vehicles of the future.

New Control System for Weather Satellite.—Tiros II, the nation's new television-infrared weather "eye" satellite, has achieved a major "first" in space with a new and simple system that enables ground observers to shift its tilt in space by remote control for improved TV picture coverage of clouds around the Earth.

The new orientation technique was developed by the Radio Corporation of America research and engineering team which designed and built both the Tiros I and Tiros II satellite-and-ground systems for the National Aeronautics and Space Administration. It uses the effect of the Earth's magnetic field in space to alter the attitude of the satellite upon command, without the need for special propulsion devices such as rocket motors or compressed gas jets. Applied for the first time in Tiros II, the system may have application to future satellite projects for stabilization and attitude control.

The new system results from studies by RCA and government scientists of an unexpected gradual shift in the attitude of the first Tiros satellite under the influence of the magnetic field surrounding the Earth. In the first Tiros, which returned nearly 23,000 useful TV cloud pictures to Earth following its launching last April 1, these magnetic forces caused the satellite to tilt gradually away from the predicted position of its axis in space.

In Tiros II, the forces are being harnessed by a technique in which a controllable magnetic field is generated around the satellite itself by a coil of wires around the lower sides of the vehicle. Interacting with the Earth's magnetic field, this gives ground observers an invisible "hand" with which to tilt the satellite on command in order to obtain a more advantageous angle for the sensors and the solar power supply. In developing the novel technique, RCA engineers devised a large spherical test "cage" in which the complete satellite was placed and rotated in a magnetic and solar environment similar to those in space.

Equipped with its new orientation-control system and with newly-developed infrared instruments to measure the emission and reflection of solar heat by the Earth and its atmosphere, Tiros II is the second step in the experimental weather satellite program being conducted by NASA to study the feasibility of regular satellite weather operations.

The first step was Tiros I, whose three months of operation during last year shed new light on global weather patterns and demonstrated the utility of satellites for weather observation. Both the Tiros I and Tiros II systems and their associated ground networks of command and receiving stations have been developed for NASA by RCA's Astro-Electronics Division, Princeton, N. J.

In the satellite itself, the major innovations of Tiros II are the orientation-control system and the infrared equipment to observe the heat balance of the Earth and its atmosphere. The infrared package, developed by NASA, includes a five-channel detector to measure selected portions of the infrared spectrum around the Earth. These provide information about the amount of reflected sunshine, the separate and total amounts of radiation from the Earth and its atmosphere, and, for reference, the amount of visible light reflected back into space.

It has been pointed out that in spite of the addition of new features to Tiros II, its total weight has been kept down to about 280 lb. as a result

of weight economies achieved in the system by RCA engineers. For example, the orientation-control system, consisting of electrical circuitry and 250 turns of wire around the outside of the satellite, weighs less than 2 lb.

In the Tiros II ground system, a major change from Tiros I is the transfer of one of the two primary ground stations from Hawaii to the Pacific Missile Range, with installations at Point Mugu, southern California, and on San Nicolas Island, 55 miles offshore. This new station, operated by the U. S. Navy for NASA, is paired with the second main ground station operated by RCA and the U. S. Army Signal Corps at Fort Monmouth, N. J. Through the two stations, Tiros II is programmed for television picture-taking and infrared measurements during the next orbit, and is commanded to "read out" the data when it comes again within range of one of the stations. This communications range is roughly 1500 miles. The main ground system is backed up with an additional ground station at RCA's Space Center at Princeton.

The remainder of the Tiros II system includes those elements which proved so successful in Tiros I. Among them are two specially-developed RCA television cameras—a wide-angle ½-in. Vidicon camera which photographs an area of about 700 miles on each side, and a narrow-angle version of the same camera which views an area about 80 miles on each side.

Associated with the cameras are two specially-designed RCA television tape recorders to store the TV pictures until the "readout" command is received from a ground station. Each of the tape recorders stores up to 32 remote TV pictures from its associated camera on each orbit. Like Tiros I, Tiros II is expected to operate for about three months, after which period its cameras will be pointing earthward in darkness. During this operating period, the satellite will complete approximately 1,300 orbits.

Among the other major features common to both the Tiros II and Tiros I satellites are electronic clocks to control the timing of operations by the cameras, tape recorders, and infrared systems during each orbit; a coating of more than 9,000 solar cells on the top and sides of the satellite to convert solar energy to electrical energy for operation of the electronic systems; "yo-yo" weights which will be used as the satellite enters orbit to slow its spin rate from 120 revolutions per minute down to 12 rpm. rotation desired for operations in orbit; and five pairs of solid fuel spin-up rockets to restore spin momentum as required during operations.

Cryogenic Memory Plane.—Successful fabrication of a cryogenic thin film memory plane the size of a postage stamp and the development of automatic control techniques capable of duplicating it many times have been announced by the International Business Machines Corporation.

This is the first time that scientists have used automatic control techniques to duplicate such a cryogenic unit. The memory plane, consisting of 135-cryotron devices built up in 19 layers of material, was fabricated by scientists at IBM's Federal Systems Division laboratory in Kingston, N. Y., working in conjunction with IBM Research.

In the device reported by IBM, three cryotrons form a memory cell which combines storage with an elementary logic operation. Because of this combination of functions,

sophisticated operations not practical before may be realized. For example, scientists expect this concept may be used in an "associative memory," which simultaneously searches all memory compartments and thereby speeds access to any stored information.

Key to this major advance in cryogenic device development is the special technique which permits accurate duplication of devices. By means of this technique, microscopically thin layers of metals and insulating materials are automatically deposited on a glass substrate.

The equipment used for deposition allows each layer of a metal or insulator to be sequentially deposited through 17 microscopically adjusted masks, or perforated metal sheets. The masks are changed automatically like records in a juke box and are held in a large metal cylinder operating under high vacuum. Once the masks have been properly aligned, the process automatically produces duplicate superconducting memory planes with similar electrical and mechanical characteristics.

Forty separate pieces or "bits" of information are stored in 120 cryotrons in the new memory plane. Of the remaining 15 cryotrons, 10 permit access to the stored bits of information; the other five are "in-line" cryotrons which switch bits of information from one memory plane to another. These newer "in-line" cryotrons operate faster than conventional cryotrons in communicating between the memory planes. They also permit faster communication between the memory planes and the arithmetic portion of the computer. Research is in progress to refine the thin-film fabrication techniques to produce cryogenic devices of even greater speeds.

Interchange of Computer Programs Using Plain English Words.—The Radio Corporation of America and Remington Rand have announced that for the first time computer programs utilizing plain English have been successfully interchanged between data processing systems of different manufacturers.

The report followed a two-day demonstration by the Remington Rand UNIVAC Division of the Sperry Rand Corporation and the RCA Electronic Data Processing Division before the Executive Committee of the Conference on Data Systems Languages (CODASYL), a government-sponsored group formed to achieve computer compatibility. The demonstration was an important milestone in the progress of data processing, since the practicability of a Common Business Oriented Language (COBOL) as an effective programming language for business-type problems was established beyond question.

In these demonstrations two operating problems, one from a government agency and one from a large industrial company, were written by RCA and Remington Rand utilizing COBOL. Each program was automatically fed into the data processing equipment of the manufacturer that prepared the program. Then, with minor changes, it was automatically prepared for the other manufacturer's equipment. In both cases the results were outstanding.

The necessity for preparing costly and time-consuming programs, or detailed instructions, for each make of computer has been a major problem facing the electronic data processing industry.

The urgency of the problem was recognized as early as May, 1959, when CODASYL was formed under the guidance of the Department of Defense. The committee consists of

government and industry representatives.

Spurred by the Defense Department's leadership, manufacturers went to work to develop a common language for computers. COBOL is the result of this research.

In the two-day demonstration, Remington Rand used the UNIVAC II at its Philadelphia Engineering Center. On the second day the conference moved to the RCA 501 Systems Center at Cherry Hill, New Jersey, where instructions were fed into the RCA computer system.

The programs were interchanged by the two companies and were fed into the computers with slight modifications. The results achieved were the same whether a UNIVAC program was being handled by the RCA system or vice-versa.

COBOL basically is a programming system that employs simple English words to instruct a computer —as opposed to the complicated machine code understood only by computer specialists.

Not only does COBOL serve to achieve compatibility between computers but the programming time can be cut in half.

A compiling system operates in this fashion: The computer is programmed to accept nouns chosen by the user that are related to the business involved—for example "payroll file," "wage," "employee number," or "tax." The computer also is programmed to accept some twenty verbs that serve as commands—"write," "read," "divide," "add," "display," or "move," along with conjunctions that link the verbs to the plain English nouns. The word "if" can tell the computer to make comparisons of data and execute logical decisions as a result of the comparisons.

The English words are transferred to magnetic tape and fed into the computer for action.

Under the COBOL System, a computer order can be written "SUBTRACT DEDUCTIONS FROM GROSS GIVING NET." Under the strictly numerical programming system, the same instruction may be fed into the computer in this form—72-010237-00-600000 and a pair of other numbers equally as long in the case of the RCA 501 or BOO549 SOO623 COO942 for the UNIVAC II.

Satellite Determination System.—A new method to determine the exact location, range, and the number of satellites orbiting the Earth, was outlined in a speech by one of the nation's foremost astronautics experts, Lt. Robert M. L. Baker Jr., at the Seventh Annual Science and Engineering Symposium, in a speech entitled, "Novel Orbit-Determination Techniques."

"It would be of military advantage," Baker said, "to include a self-contained, friend-or-foe satellite identification device on board a Satellite Inspector (SAINT)-type vehicle—a device that would not be dependent upon a continuous communication link with either friendly stations on Earth or other satellite stations.

"Such a device," Baker continued, "might be based upon the unique 'license plate' carried around by every space vehicle—that is, its orbital elements.

"The determination of satellite orbits from satellite observations," Baker added, "offers significant advantages over terrestrial orbit determination."

Baker outlined these advantages:

First there would be no ionospheric refraction problems.

Second, radar frequencies could be chosen that would greatly handicap a terrestrial station from moni-

toring the active orbit-determination procedure.

Third, knowledge could be maintained of launches and space vehicle maneuvers anywhere on Earth.

"From this point of view," Baker concluded, "the data collected by these 'space reconnaissance' or 'inspection' satellites could be utilized directly to augment the capability of the National Space Surveillance System as well as provide our satellite systems with an immediate reaction capability."

Lens-Studded Satellites Aid World-Wide Communications.—The U. S. Air Force may depend upon lens-studded satellites, now under development at the Air Research and Development Command's Wright Air Development Division, Wright-Patterson AFB, Dayton, Ohio as relay links in a world-wide communications network.

The satellite is covered with radio frequency lenses mounted over spherical reflectors. It is designed so that no matter which position the satellite assumes with regard to the Earth, strong reflection of signals will take place between ground stations 6000 miles apart.

The lens-reflector arrangement achieves a high degree of reflectivity with a relatively small satellite. The lenses focus the radio frequency energy at a point on the reflector, and again focus the reflected energy into a beam encompassing the Earth.

This offers the advantage over Echo-type satellites of directing all reflected radio energy back toward the Earth. Without a system to steer the signals earthward, much of the energy would be scattered into space.

A lens-reflector satellite the same size as the current "satelloon" (100 ft. in diameter) will reflect signals 100 times more powerfully than does Echo, according to WADD scientists. They also theorize that spinning or tumbling of the satellite will not impair its efficiency.

First test model of the satellite will be about 10 ft. in diameter and will be suspended from a high-altitude balloon for its initial tests.

Development of this satellite is part of the Air Force's passive communications satellite program, which is being directed by ARDC's Air Force Command and Control Development Division (L. G. Hanscom Field, Bedford, Mass.).

Passive satellites offer many advantages over communications satellites equipped with electronic repeating devices. Useful lifetime of passive satellites is nearly indefinite, depending only upon the orbit attained, and passive vehicles also have a much higher degree of reliability since the communications electronics are all ground-based and are readily accessible for maintenance.

The passive satellites also have relatively low development and production costs, can handle large bandwidths, and reach operational availability rapidly. Further, electronic countermeasures (jamming) must be directed against ground equipment, rather than satellite-carried instruments. In ground-based communications equipment, size, weight, and power requirements are not critical.

Unlike the Echo satelloon, the WADD satellite will be made rigid through a system of chemically-inflated ribs. This is necessary to prevent deformation of the satellite and a resulting unplanned scattering of reflected radio energy.

Explosive Gas Detector.—A new instrument for the detection of dangerous or explosive vapor mixtures has been introduced by the Instru-

ments and Systems Section of the Industrial Equipment Division of Engelhard Industries, Inc., of Newark, N. J. Called "the Bilge Bug," the device is primarily intended for use aboard boats but may also be used advantageously in industrial or other locations to detect concentrations of gasoline, natural gas and many other combustible vapors.

Consisting of a control and indicator panel, a sensor unit and a connecting cable assembly, the Bilge Bug is supplied for two operating voltages of 6 and 12 volts d.c., and draws a current of only 1.2 amp. The sensor unit is installed in any location where combustible mixtures may collect and the control unit is mounted in some convenient location such as the helm of a boat.

In operation, the switch is turned on, and a pilot light indicates proper operation and illuminates the face of the meter. Within seconds, the two platinum wire sensing elements will react to the presence of vapors and indicate the condition on the meter. Following the check, if the reading is "safe," the instrument may be turned off and the engine started.

Self-Extinguishing Epoxies.—Design and production of two self-extinguishing epoxy resins are announced by The Dow Chemical Company of Midland, Mich. Both resins are made self-extinguishing through halogen—bromine—substituted on the epoxy molecule. They are cured or hardened with the common epoxy curing agents.

According to Dow, self-extinguishing epoxy resins should find use in laminated aircraft structures, glass and paper-based laminated electrical circuits, filled castings, and in aircraft and missile adhesives. Marine paints, architectural coatings and sheath coatings (for solid or foamed plastics) are other likely uses.

Experimental Resin X-3442 is a semi-solid containing about 49 per cent bromine and has utility in blends with conventional liquid epoxy resins. Experimental Resin X-3441.1 is a solid resin containing about 19 per cent bromine and can be used either alone or in blends with conventional solid resins. A bromine content of 15–25 per cent is said to make even clear castings self-extinguishing.

Physical and electrical properties of castings and laminates made with the new resins are comparable to properties developed by conventional epoxy resins. Modulus properties are reduced at 300°F., but blends of self-extinguishing epoxy with Dow's epoxy novolac resin are expected by the company to result in higher modulus at elevated temperatures.

Prepreg laminates made with either of the self-extinguishing resins meet the strict NEMA G-11 physical and electrical properties except for 300°F. flexural strength. The resins are therefore suitable for use in many commercial applications.

"Talking" Through Touch.—A research psychologist has revealed the feasibility of giving pilots another channel of communications, for in these days of supersonic speeds, eyes and ears cannot cope with the mass of communications fed into a pilot's brain. Why not, research scientists reasoned, give pilots a third channel of communications?

Under the direction of Psychologist Richard A. Newman, the Human Engineering Laboratory of the Rome Air Development Center (RADC) undertook the research program to add another communication channel to the brain. RADC is a sub-unit

of the Air Research and Development Command.

The program was revealed in a paper prepared by Newman for delivery at ARDC's Seventh Annual Science and Engineering Symposium.

The new method involves change of vibrations, or frequencies of the voice, into mechanical vibrations which a pilot can feel by placing a small plate in contact with the skin.

Actually, the plate could be placed anywhere on the body. But research revealed that an area on the pilot's chest is more receptive to pulses than other parts of the body.

Newman emphasized that the project still is strictly in the research stage. He added, however, he felt scientists were making substantial progress with the technique.

"So far as application for the Air Force is concerned," Newman said, "we are thinking of developing the touch system for channeling information to a pilot when his hearing or sight has been impaired by injury, weather, or environmental conditions."

Scientists envision numerous conditions under which the technique might prove practical.

On takeoff, for instance, jet pilots perform many manual functions while visually checking numerous instruments and talking with the control tower operator. Use of a small vibrator on the body would give them another means of receiving certain types of information.

The psychologist pointed out that the system could be used as a back-up channel when jet noise or other technical difficulties disrupt normal communications.

Another situation when this means of communication might prove practical would be individuals responsible for missile firings. As the moment of launch approaches, numerous individuals must perform multiple tasks almost simultaneously. A third communications channel would enable them to perform more efficiently.

A third use of the "sensor" might be when an individual would want to communicate with another without the knowledge of persons nearby. A small vibrator attached to the body could easily go unnoticed.

"As in other Air Force research projects, this system may have far-reaching civilian applications," Newman said.

In the future, touch communications might be used to assist deaf people to "hear" through their skin. This might be done by training them to feel the vibrations from voices by using a small vibrator.

Newman says the technique offers much promise to individuals associated with the aerospace age. He added, however, with typical scientific caution, that "a more extensive research program will be required before a fully operable technique can be developed."